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The First Order Shear Deformation Theory for Sandwich Plates

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Summary

This paper presents the specifics of applying the First Order Shear Deformation Theory (FSDT) for determining the bending response parameters of sandwich plates. The classical lamination theory (CLT) is used to establish the stiffness matrices of the element, the significance and importance of the shear correction factor, and finally, different methods (analytical and numerical) for implementing the theory are presented.

KEYWORDS: sandwich plates, first order shear deformation theory, bending

1. INTRODUCTION

Sandwich plates are frequently used because of their ability to provide high bending stiffness while being light weight. A sandwich panel can be assimilated with a I-beam in which the faces act as the flanges who carry the normal stresses caused by bending moments and the core as the web supporting the shear stresses caused by transverse forces. However, typically the core is made of materials with reduced stiffness, which results in the appearance of shear effects that need to be accounted for.

The first order shear deformation theory (FSDT), commonly referred to as the Mindlin-Reissner theory, is the most basic tool available to take into account such effects. The theory is based on the following displacement field:

$$\begin{aligned} u(x,y,z) &= z\phi_x(x,y) \\ v(x,y,z) &= z\phi_y(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{1}$$

where Φ_x and Φ_y denote rotations about the x and y axes, respectively. The rotations Φ_x and Φ_y are no longer explicit functions of the derivatives of the deflection w_0, as for the classical plate theory.

From the displacement field (1), the components of the linear strains are (fig. 1):

$$\varepsilon_x = z\frac{\partial \phi_x}{\partial x};\quad \varepsilon_y = z\frac{\partial \phi_y}{\partial y};\quad \gamma_{xy} = z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$\gamma_{xz} = \phi_x + \frac{\partial w_0}{\partial x};\quad \gamma_{yz} = \phi_y + \frac{\partial w_0}{\partial y} \tag{2}$$

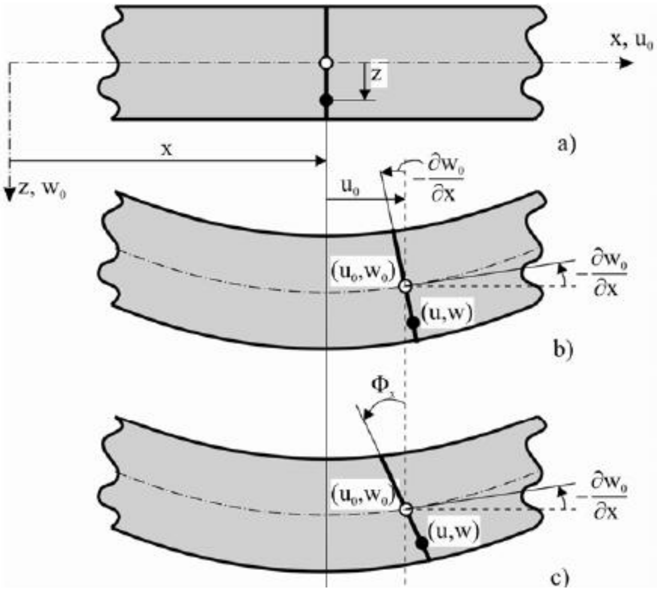

Figure 1. Undeformed and deformed geometries of an edge plate:
a) undeformed; b) Kirchoff plate theory; c) Mindlin-Reissner plate theory

The kinematics of FSDT assume a global transverse shear strain considered constant on the plate thickness. To compensate for this assumption, a shear correction factor is required, which appears as a coefficient in the expression for the transverse shear stress resultant. The accuracy with which this factor is computed is paramount for the validation of the results.

2. ANALYSIS OF SANDWICH PLATES USING FSDT

2.1. Laminate stiffness's

In order to establish the constitutive equations for sandwich plates and determine the necessary parameters to accurately estimate the bending behaviour of said plates, it is necessary to employ the Classical Lamination Theory (CLT) to determine the stiffness matrices (extensional stiffness matrix – [A], coupling

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matrix [B] and bending stiffness matrix – [D]). For simplicity, this papers strictly refers to rectangular isotropic sandwich plates with equal face sheets, hence the [B] matrix will be null. According to [3], a sandwich plate is isotropic when the core is made of an isotropic (such as foam) or transversely isotropic (such as honeycomb) material and the top and bottom facesheets are made of identical isotropic materials or are identical quasi-isotropic laminates.

There are two ways to approach the problem:

1) If the core is made of a material with a much lower modulus of elasticity than that of the faces, then its influence on the overall stiffness of the plate can be completely neglected, and the method presented in 2.1.1 can be used;

2) If the core material and that of the faces both have comparable moduli of elasticity, then the method presented in 2.1.2 must be used.

2.1.1. The stiffness matrices for the case when $E_c << E_f$

First, the face stiffness's are determined as follows [3]:

$$A_f = 2 \cdot h_f \cdot \frac{E_f}{1-\nu_f^2}; \quad D_f = E_f \cdot \frac{2 \cdot \left(d^2 \cdot h_f + \frac{h_f^3}{12} \right)}{1-\nu_f^2} \tag{3}$$

where, according to Figure 2:

- d represents the distance from the centre of the plate to the mid-plane of the faces;
- h_f and h_c are the thickness of the faces and of the core;
- E_f and ν_f are the Young modulus and Poisson ratio for the facesheets material.

Next, the stiffness matrices for the whole panel can be determined [3]:

$$A = A_f \cdot \begin{bmatrix} 1 & \nu_f & 0 \\ \nu_f & 1 & 0 \\ 0 & 0 & \frac{1-\nu_f}{2} \end{bmatrix}; \quad D = D_f \cdot \begin{bmatrix} 1 & \nu_f & 0 \\ \nu_f & 1 & 0 \\ 0 & 0 & \frac{1-\nu_f}{2} \end{bmatrix} \tag{4}$$

2.1.2. The stiffness matrices for the case when E_c is comparable to E_f

In this situation, the components of the stiffness matrices can be determined as follows:

$$A_{ij}=\sum_{k=1}^{3}(Q_{ij})_k(z_k-z_{k=1})\;;\quad D_{ij}=\sum_{k=1}^{3}(Q_{ij})_k(z_k^3-z_{k=1}^3)\;;\quad i,j=1,2,6 \tag{5}$$

$$A_{44}=A_{55}=2\cdot G_f\cdot h_f+G_c\cdot h_c \tag{6}$$

where $(Q_{ij})_k$ are terms of the plane stress-reduced elastic coefficient matrices for the materials of the core and faces of the plate and can be determined like so:

$$Q_{11}=Q_{22}=\frac{E}{1-\nu^2}\;;\quad Q_{12}=Q_{21}=\frac{\nu\cdot E}{1-\nu^2}\quad Q_{66}=G \tag{7}$$

with *E*, *G* and *ν* - the Young modulus, shear modulus and Poisson's ratio of the materials of the core (*c* index) and faces (*f* index) of the sandwich panel.

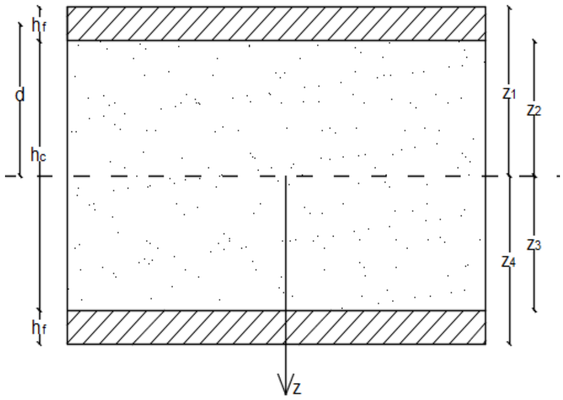

Figure 2. Distances required for determining the stiffness matrices

2.2. The shear correction factor

Since the transverse shear strains are represented as constant through the laminate thickness, consequently, the transverse shear stresses will also be constant. It is well known that the transverse shear stresses vary parabolically. The discrepancy between the actual stress state and the constant stress state predicted by FSDT is corrected by multiplying the transverse shear forces with a coefficient K_s, called *shear correction factor*.

For elements with homogenous cross-sections, the value for the shear correction factor, accepted in the literature [6], is 5/6. However, for composite laminates, this value is no longer suitable. In [4, 8], the authors present a series of methods to determine this factor.

The most widely used method for computing the shear correction factor is the so called energy equivalence method. The strain energy due to transverse shear stresses predicted by FSDT is equalled with that from the three-dimensional elasticity theory.

For example, for a symmetric sandwich beam, the shear correction factor can be determined like so:

$$K_s = \frac{D^2}{A_{55} \cdot (2 \cdot b_1 + b_2)} \tag{8}$$

where:

$$D = \frac{E_f \cdot (h^3 - h_c^3) + E_c \cdot h_c^3}{12}$$

$$b_1 = \frac{E_f^2}{4G_f} \cdot \left[\left(\frac{8}{15}\right)\left(\frac{h}{2}\right)^5 - \left(\frac{h}{2}\right)^4 \left(\frac{h_c}{2}\right) + \left(\frac{2}{3}\right)\left(\frac{h}{2}\right)^2 \left(\frac{h_c}{2}\right)^3 - \left(\frac{h_c^5}{2^5 \cdot 5}\right) \right] \tag{9}$$

$$b_2 = \frac{\left\{ E_c^2 \cdot \frac{h_c^5}{2^5 \cdot 5} + E_c \cdot \left(\frac{h_c}{2}\right)^3 \frac{E_f (h_c^2 - h^2) - E_c h_c^2}{6} + \left[E_f (h_c^2 - h^2) - E_c h_c^2\right]^2 \frac{h_c}{32} \right\}}{2 \cdot G_c}$$

In [4], Birmann and Bert concludes that this method is not reliable if the modular ratio for the materials of the faces and core is extremely high, in which case the values predicted for the shear correction factor are close to zero. This statement has also been proven in [9], where the authors studied the bending behaviour of a sandwich plate with the core made of extruded polystyrene and aluminium faces. Because of the low values of $K_{s,}$ the plate deflections predicted by FSDT were several orders of magnitude greater than those of CLPT.

Birmann and Bert suggests that for the design of sandwich plates, the shear correction value should be taken equal to unity.

2.3. Bending analysis of sandwich plates using FSDT

2.3.1. The governing differential equations

The governing differential equations for a laminated composite plate subjected to a transverse load (Figure 3) are [7]:

$$D_{11} \frac{\partial^2 \phi_x}{\partial x^2} + D_{66} \frac{\partial^2 \phi_x}{\partial y^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_y}{\partial x \partial y} - K_s A_{55} \left(\phi_x + \frac{\partial w}{\partial x} \right) = 0$$

$$(D_{12}+D_{66})\frac{\partial^2\phi_x}{\partial x\partial y}+D_{66}\frac{\partial^2\phi_y}{\partial y^2}+D_{22}\frac{\partial^2\phi_y}{\partial y^2}-K_sA_{44}\left(\phi_y+\frac{\partial w}{\partial y}\right)=0$$
$$K_sA_{55}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+K_sA_{44}\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+q(x,y)=0 \quad (10)$$

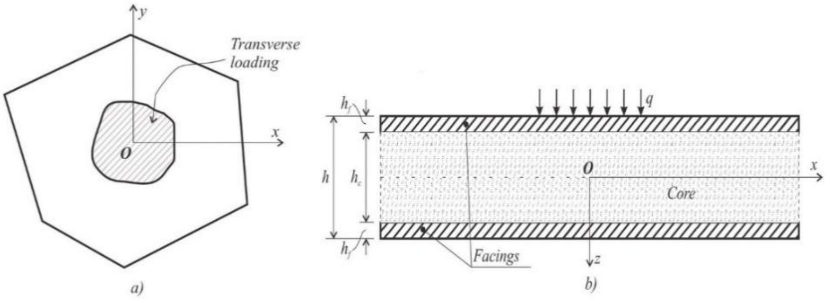

Figure 3. Geometry and loading of a sandwich plate: a) general polygonal shape; b) plate cross section

The inclusion of transverse shear deformation effects results in three coupled partial differential equations with three unknowns, Φ_x, Φ_y and w, as opposed to having one partial differential equation with one unknown, w, in classical plate theory.

The plate constitutive equations for the classical or first-order shear deformation theories are:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}=\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{11} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}\begin{Bmatrix} \dfrac{\partial\phi_x}{\partial x} \\ \dfrac{\partial\phi_y}{\partial y} \\ \dfrac{\partial\phi_x}{\partial y}+\dfrac{\partial\phi_y}{\partial x} \end{Bmatrix} \quad (11)$$

$$\begin{Bmatrix} V_y \\ V_x \end{Bmatrix}=K_s\begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix}\begin{Bmatrix} \phi_y+\dfrac{\partial w_0}{\partial y} \\ \phi_x+\dfrac{\partial w_0}{\partial x} \end{Bmatrix} \quad (12)$$

where the laminate stiffness's D_{ij} and A_{ij} are defined in [1, 2, 5, 7] for symmetric laminates with multiple isotropic layers.

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2.3.2. *The Navier Solution for FSDT*

The boundary conditions for a simply supported plate (Figure 4), in FSDT, are:

- for the edge parallel to the y-axis: $v_0 = w_0 = \phi_y = M_x = 0$
- for the edge parallel to the x-axis: $u_0 = w_0 = \phi_x = M_y = 0$

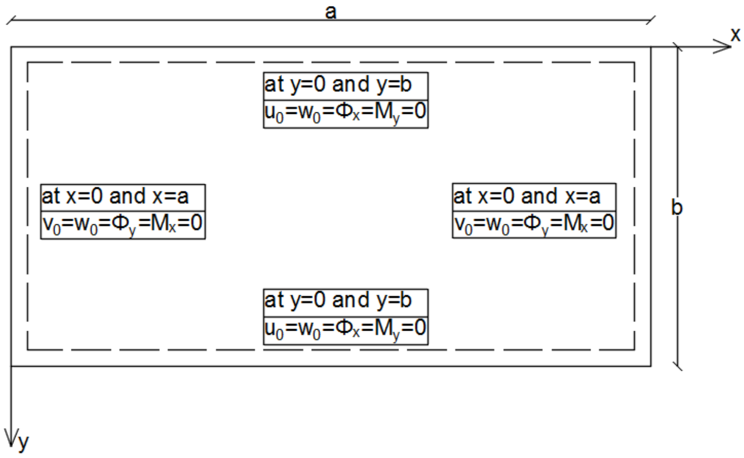

Figure 4. Boundary conditions on a simply supported plate for the Mindlin theory

The boundary conditions are satisfied by the following expressions:

$$\begin{aligned} w_0(x,y) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn} \sin\alpha x \sin\beta y \\ \phi_x(x,y) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_{mn} \cos\alpha x \sin\beta y \\ \phi_y(x,y) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Y_{mn} \sin\alpha x \cos\beta y \end{aligned} \tag{13}$$

where $\alpha = m\pi/a$, $\beta = n\pi/b$ and *m*, *n* are the steps of the series.

The transverse load is also transcribed in double Fourier series:

$$q(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Q_{mn} \sin\alpha x \sin\beta y \tag{14}$$

$$Q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\cdot\sin\alpha x\cdot\sin\beta y\cdot dxdy \tag{15}$$

The coefficients W_{mn}, X_{mn} and Y_{mn} have the following expressions:

$$W_{mn}=\frac{b_0}{b_{mn}}Q_{mn}\;;\quad X_{mn}=\frac{b_1}{b_{mn}}Q_{mn}\;;\quad Y_{mn}=\frac{b_2}{b_{mn}}Q_{mn} \tag{16}$$

where:

$$b_0=s_{22}\cdot s_{33}-s_{23}\cdot s_{23};\quad b_1=s_{23}\cdot s_{13}-s_{12}\cdot s_{33};\quad b_2=s_{12}\cdot s_{23}-s_{22}\cdot s_{13}$$
$$b_{mn}=s_{11}\cdot b_0+s_{12}\cdot b_1+s_{13}\cdot b_2 \tag{17}$$

and

$$s_{11}=K_s(A_{55}\cdot\alpha^2+A_{44}\cdot\beta^2);\quad s_{22}=(D_{11}\cdot\alpha^2+D_{66}\cdot\beta^2+K_sA_{55})$$
$$s_{12}=K_s\cdot A_{55}\cdot\alpha;\quad s_{23}=(D_{12}+D_{66})\cdot\alpha\cdot\beta \tag{18}$$
$$s_{13}=K_s\cdot A_{44}\cdot\beta;\quad s_{33}=(D_{66}\cdot\alpha^2+D_{22}\cdot\beta^2+K_sA_{44})$$

Once the displacements and rotations are determined, the bending and twisting moments can be computed:

$$M_x=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(D_{11}\cdot\alpha\cdot X_{mn}+D_{12}\cdot\beta\cdot Y_{mn})\cdot\sin\alpha x\sin\beta y$$
$$M_y=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(D_{12}\cdot\alpha\cdot X_{mn}+D_{22}\cdot\beta\cdot Y_{mn})\cdot\sin\alpha x\sin\beta y \tag{19}$$
$$M_{xy}=D_{66}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\beta\cdot X_{mn}+\alpha\cdot Y_{mn})\cdot\cos\alpha x\cos\beta y$$

and the stresses:

$$\sigma_x=-z\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(Q_{11}\cdot\alpha\cdot X_{mn}+Q_{12}\cdot\beta\cdot Y_{mn})\cdot\sin\alpha x\sin\beta y$$
$$\sigma_y=-z\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(Q_{12}\cdot\alpha\cdot X_{mn}+Q_{22}\cdot\beta\cdot Y_{mn})\cdot\sin\alpha x\sin\beta y$$
$$\tau_{xy}=-z\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}-Q_{66}(\beta\cdot X_{mn}+\alpha\cdot Y_{mn})\cdot\cos\alpha x\cos\beta y \tag{20}$$
$$\tau_{yz}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}G_c(Y_{mn}+\beta\cdot W_{mn})\cdot\sin\alpha x\cos\beta y$$
$$\tau_{xz}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}G_c(X_{mn}+\alpha\cdot Y_{mn})\cdot\cos\alpha x\sin\beta y$$

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2.3.3. Numerical methods

2.3.3.1. Finite difference method

The finite difference method can easily be used to determine the bending behaviour of isotropic sandwich plates for the first order shear deformation theory.

The first step consists in determining the moment sum and deflections for an equivalent homogenous plate, composed of the two face sheets glued together, using the well-known Poisson equations of the Classical Plate Theory:

$$\nabla^2 M^K = -q\,; \quad \nabla^2 w^K = -\frac{M^K}{D} \tag{21}$$

After which the deflections of the real sandwich plate are computed using the relationship that connects the two theories [5]:

$$w^M = \frac{D}{D_c + D_f} w^K + \frac{M^K}{K_s (G_c h_c + 2G_f h_f)} \tag{22}$$

where:

- ∇^2 - the Laplacian operator, approximated by the finite difference method;
- q is the transverse load;
- "M" and "K" superscripts denote quantities of the Kirchoff and Mindlin plate theories;
- D is the bending stiffness of the equivalent homogenous plate:

$$D = \frac{E_f \cdot (2h_f)^3}{12 \cdot (1-\nu_f^2)} \tag{23}$$

- M^K is the moment sum for the homogenous equivalent Kirchoff plate;
- D_c and D_f are the bending stiffnesses of the core and faces for the real plate;
- G_c, G_f - the shear moduli of the materials for the core and faces;
- h_c and h_f - the thickness of the core and that of the faces;
- K_s – the shear correction factor, as discussed in section 2.2.

The stiffnesses D_c and D_f are:

$$D_c = \frac{E_c h_c^3}{12(1-\nu_c^2)}\,; \quad D_f = \frac{2E_f h_f \left(\frac{3h_c^2}{4} + \frac{3h_c h_f}{2} + h_f^2 \right)}{3(1-\nu_f^2)} \tag{24}$$

2.3.3.2. Finite elements method

Almost all commercial finite element analysis codes (Abaqus, Ansys) use Mindlin type plate elements.

For example, the most common type of element used by Ansys to model layered plates is SHELL181. The following is from the Ansys Workbench Help [10]:

“SHELL181 is a four-node element with six degrees of freedom at each node: translations in the x, y and z directions, and rotations about the x, y and z-axes. SHELL181 can be used for layered applications for modelling composite shells or sandwich construction. The accuracy in modelling composite shells is governed by the first-order shear-deformation theory.(usually referred to as Mindlin-Reissner theory).

Transverse shear stiffness of the shell section is estimated by an energy equivalence procedure. The accuracy of this calculation may be adversely affected if the ratio of material stiffness’s (Young’s moduli) between adjacent layers is very high.”

3. CONCLUSIONS

The first-order shear-deformation theory is widely used for the bending analysis of sandwich plates. It relaxes the normal segment hypothesis and takes into account a constant shear strain on the plate thickness. In order to correct the discrepancy between this constant distribution and the real parabolic distribution, the theory uses a shear correction factor. As has been shown, the accuracy with which this factor is computed is critical for the validity of the results. The Mindlin plate model is implemented in all major finite element analysis software and the shear correction factors are determined using the energy equivalence method.

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