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EBBEF2p - A Computer Code for Analysing Beams on Elastic Foundations

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Summary

The development of a finite element computer code for the static structural analysis of beams on elastic foundation is described. Called EBBEF2p (Euler-Bernoulli Beams on Two-Parameter Elastic Foundationis) this code is written in the computer programme package MATLAB and can handle a wide range of static loading problems involving a one-dimensional beam supported by elastic foundation. The theoretical basis for the code, its computer implementation, and its use to solve example problems are discussed too.

KEYWORDS: Beams; Elastic Foundations; Winkler Foundation; Two-Parameter Elastic Foundation; Vlasov Foundation; Finite Elements; Static Structural Analysis; Computer Program.

1. INTRODUCTION

The concept of beams on elastic foundations it is extensively used by geotechnical, pavement and railroad engineers for foundation design and analysis.

Currently, the analysis of beams on elastic foundation is performed by using special computer programs based on numerical methods, such as Finite Difference Method (FDM) and Finite Element Method (FEM). However, these programs are limited in their application, most of them being developed only for a very simple subgrade model, Winkler's Hypothesis. They cannot be used for other soil models such as Two-Parameter, Elastic Half-Space or Elastic Layer and others.

This paper describes a finite element computer program, as a toolbox to MATLAB, developed to analyse the interaction between a beam and its two-parameter elastic foundation. By considering a linear variation of both foundation parameter, EBBEF2p can account in a consistent way for the bearing soil inhomogeneity. It can be used for any practical static loading and support condition including prescribed displacement.

The numerical model uses a cubic Hermitian polynomial to interpolate nodal values of the displacements field for a two-node beam elements. The elemental stiffness matrix and load vector are obtained by using Galerkin's Residual Method

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with adding the contribution of the foundation as element foundation stiffness matrices to the regular flexure beam element.

2. BEAM-SOIL SYSTEM MODELLING

Generally, the analysis of bending of beams on an elastic foundation is developed on the assumption that the reaction forces of the foundation are proportional at every point to the deflection of the beam at that point. The vertical deformation characteristics of the foundation are defined by means of identical, independent, closely spaced, discrete and linearly elastic springs. The constant of proportionality of these springs is known as the modulus of subgrade reaction, k_s. This simple and relatively crude mechanical representation of soil foundation was first introduced by Winkler, in 1867 [1], [2].

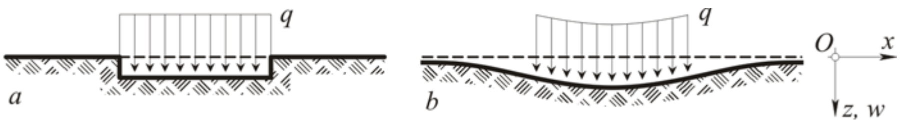

Figure 1. Deflections of elastic foundations under uniform pressure: *a* – Winkler foundation; *b* – practical soil foundations.

The Winkler model, which has been originally developed for the analysis of railroad tracks, is very simple but does not accurately represents the characteristics of many practical foundations. One of the most important deficiencies of the Winkler model is that a displacement discontinuity appears between the loaded and the unloaded part of the foundation surface. In reality, the soil surface does not show any discontinuity (Figure 1).

Historically, the traditional way to overcome the deficiency of Winkler model is by introducing some kind of interaction between the independent springs by visualising various types of interconnections such as flexural elements (beams in one-dimension (1-D), plates in 2-D), shear-only layers and deformed, pretensioned membranes [1]. The foundation model proposed by Filonenko and Borodich in 1940 [1] acquires continuity between the individual spring elements in the Winkler model by connecting them to a thin elastic membrane under a constant tension. In the model proposed by Hetényi in 1950 [1], interaction between the independent spring elements is accomplished by incorporating an elastic plate in three-dimensional problems, or an elastic beam in two-dimensional problems, that can deforms only in bending. Another foundation model proposed by Pasternak in 1954 acquires shear interaction between springs by connecting the ends of the springs to a layer consisting of incompressible vertical elements which deform only by transverse shearing [1].

This class of mathematical models have another constant parameter which characterizes the interaction implied between springs and hence are called *two-parameter models* or, more simply, *mechanical models* (Figure 2).

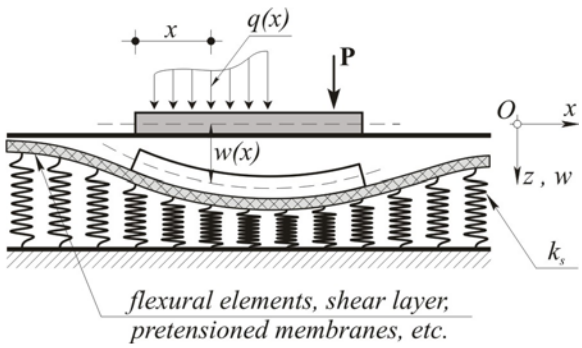

Figure 2. Beam resting on two-parameter elastic foundation.

Another approach to developing, and also to improve foundation models, starts with the three complex sets of partial-differential equations (compatibility, constitutive, equilibrium) governing the behavior of the soil as a semi-infinite continuum, and then introduce simplifying assumptions with respect to displacements or/and stresses in order to render the remaining equations fairly easy to solve in an exact, closed-form manner. These are referred to as *simplified-continuum models*.

Vlasov, in 1960, adopted the simplified-continuum approach based on the variational principle and derived a two parameter foundation model [3]. In his method the foundation was treated as an elastic layer and the constraints were imposed by restricting the deflection within the foundation to an appropriate mode shape, $\phi(z)$. The two parameter Vlasov model (Figure 3) accounts for the effect of the neglected shear strain energy in the soil and shear forces that come from surrounding soil by introducing an arbitrary parameter, γ, to characterize the vertical distribution of the deformation in the subsoil [3]; the authors did not provide any mechanism for the calculation of γ. Jones and Xenophontos [3] established a relationship between the parameter γ and the displacement characteristics, but did not suggest any method for the calculation of its actual value. Following Jones and Xenophontos, Vallabhan and Das [4] determined the parameter γ as a function of the characteristic of the beam and the foundation, using an iterative procedure. They named this model a *modifed Vlasov model* [4], [5].

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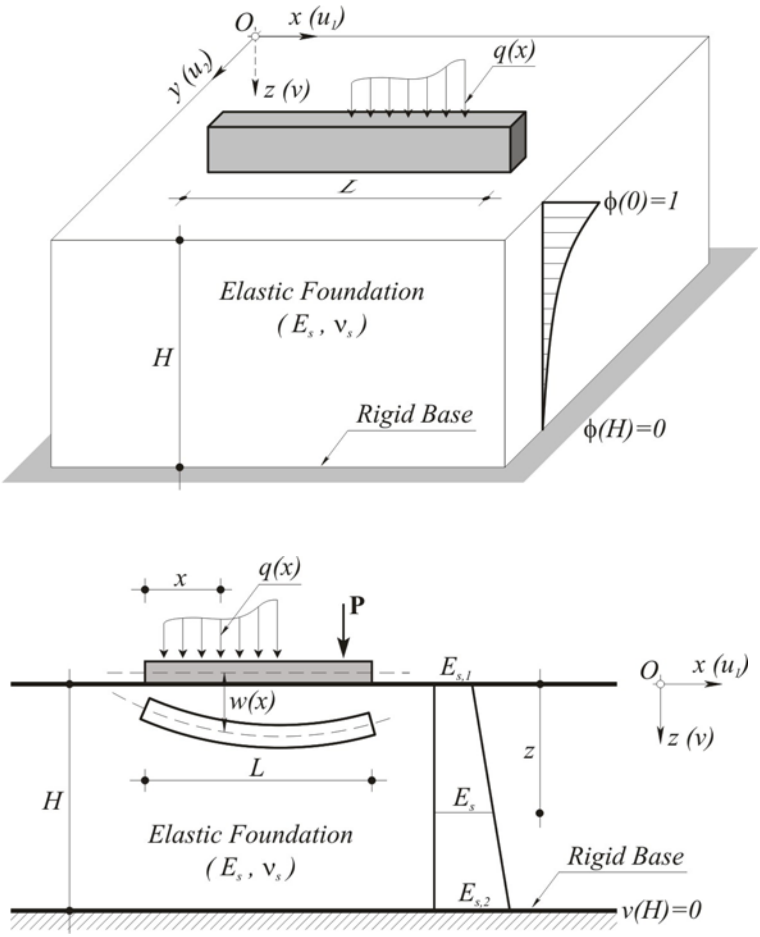

Figure 3. Beam resting on two-parameter Vlasov foundation.

3. FINITE ELEMENT MODELLING

All foundation models shown foregoing lead to the same differential equation. Basically, all these models are equivalent and differ only in the definition of its parameters [6].

3.1. Governing Differential Equation

The various two-parameter elastic foundation models define the reactive pressure of the foundation $p(x)$, as [6]

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$$p(x) = k_s B w(x) - k_1 B \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = k w(x) - \bar{k}_1 \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \tag{1}$$

where: B is the width of the beam cross section; w – deflection of the centroidal line of the beam and k_1 is the second foundation parameter with a different definition for each foundation model. As a special case, if the second parameter k_1 is neglected, the mechanical modelling of the foundation converges to the Winkler formulation. For the case of a (linear) variable subgrade coefficients, Equation (1) may be written as

$$\begin{aligned} p(x) &= k_s(x) B w(x) - k_1(x) B \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = \\ &= k(x) w(x) - \bar{k}_1(x) \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} \end{aligned}. \tag{2}$$

Using the last relation and beam theory, one can generate the governing differential equations for the centroidal line of the deformed beam resting on two-parameter elastic foundation as [6]

$$EI \frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} + k(x) w(x) - \bar{k}_1(x) \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = q(x), \tag{3}$$

where: E is the modulus of elasticity for the constitutive material of the beam; I – the moment of inertia for the cross section of the beam and $q(x)$ is the distributed load on the beam.

3.1.1. Parameters estimation

It is difficult to interpret exactly what subgrade material properties or characteristics are reflected in the various mechanical elements (springs, shears layers, etc.) thus evaluation on a rational, theoretical basis is cumberstome. The advantage of simplified-continuum approach is the elimination of the necesity to determine the values of the foundation parameters, arbitrarily, because these values can be computed from the material properties (deformation modulus E_s, Poisson number v_s and depth of influence zone H along the beam) for the soil. Thus there is insight into exactly what each model assumes and implies in terms of subgrade behavior.

With the assumptions of

vertical displacement,

$$v(x,z) = w(x)\phi(z) \text{ and} \tag{4}$$

horizontal displacement,

$$u_1(x,z)=0\,, \tag{5}$$

and using variational calculus, Vlasov model parameters are expressed as [3], [4], [5]

$$k_s=\int_0^H \frac{E_s(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)}\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2\mathrm{d}z\,,\quad k_1=\int_0^H \frac{E_s}{2(1+\nu_s)}\phi^2\,\mathrm{d}z\,, \tag{6}$$

were

$$\phi(z)=\frac{\sinh\gamma\left(1-\frac{z}{H}\right)}{\sinh\gamma}\,, \tag{7}$$

is a function defining the variation of the deflection $v(x, z)$ in the z direction, which satisfy the boundary condition shown in Figure 3, and

$$\left(\frac{\gamma}{H}\right)^2=\frac{1-2\nu_s}{2(1-\nu_s)}\frac{\int_{-\infty}^{+\infty}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\mathrm{d}x}{\int_{-\infty}^{+\infty}w^2\,\mathrm{d}x}\,. \tag{8}$$

Since γ is not known apriory, the solution technique for parameters evaluation is an iterative process which is dependent upon the value of the parameter γ. Therefore, by assuming an approximate value of γ initially, the values of k and k_1 are evaluated using (6). From the solution of the deflection of the beam, the value of γ is computed using (8). The new γ value is again used to compute new values of k and k_1. The procedure is repeated until two succesive values of γ are approximately equal [4], [5].

3.2. Finite Element Formulation

The assumptions and restrictions underlying the development are the same as those of elementary beam theory with the addition of [7], [8]

- the element is of length l and has two nodes, one at each end;
- the element is connected to other elements only at the nodes;
- element loading occurs only at the nodes.

Figure 4. shows a finite element of a beam on a two-parameter elastic foundation were $\{d_e\}=\{w_1\ \theta_1\ w_2\ \theta_2\}^T$ are the degrees of freedom of the element and $\{S_e\}=\{Q_1\ M_1\ Q_2\ M_2\}^T$ are loads applied to the nodes [7], [8].

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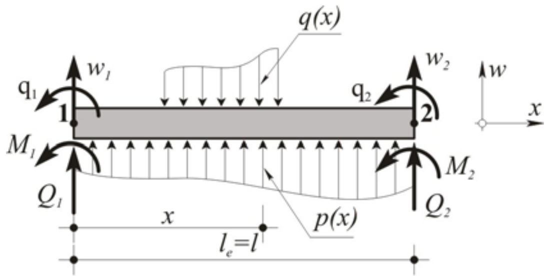

Figure 4. Nodal degrees of freedom and corresponding nodal forces on a beam element on a two-parameter foundation.

It must note that Q_1 and Q_2 from the load vector $\{S_e\}$ are not simply the transverse shear forces in the beam; they includes also the shear resistance associated with modulus of the two-parameter foundation [6]. Force Q_i (i = 1, 2), is a generalized shear force defined by

$$Q_i = V_i + V_i^*, \tag{9}$$

were: $V_i = EI\dfrac{\mathrm{d}^3 w(x)}{\mathrm{d}x^3}$ is the usual shear contribution from elementary beam theory; $V_i^* = -\bar{k}_1 \dfrac{\mathrm{d}w(x)}{\mathrm{d}x}$ – the shear contribution from two-parameter elastic foundation (negative sign arises because a positive slope requires opposite shear forces in the foundation) [6].

Considering the four boundary conditions and the one-dimensional nature of the problem in terms of the independent variable, we assume the displacement function in the form [7], [8]:

$$w_e(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \tag{10}$$

For the both foundation parameters a linear variation is considered [8],

$$k(x) = k_{(1)} + \frac{k_{(2)} - k_{(1)}}{l} x, \quad \bar{k}_1(x) = \bar{k}_{1,(1)} + \frac{\bar{k}_{1,(2)} - \bar{k}_{1,(1)}}{l} x. \tag{11}$$

The choice of a cubic function to describe the displacement is not arbitrary. With the specification of four boundary conditions, we can determine no more than four constants in the assumed displacement function. The second derivative of the assumed displacement function, $w_e(x)$ is linear; hence, the bending moment varies linearly, at most, along the length of the element. This is in accord with the assumption that loads are applied only at the element nodes [7].

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Applying the boundary conditions and solving the constants from (10) and then substituting the results back into (10) we can obtain the interpolation form of the displacement as [7], [8]

$$w_e(x) = N_1(x)w_1 + N_2(x)\theta_1 + N_3(x)w_2 + N_4(x)\theta_2 = [N_i]^T\{d_e\}, \tag{12}$$

where $N_i(x)$, $(i = 1, \ldots, 4)$ are the well-known shape functions of Hermite type that describe the distribution of displacement in terms of nodal values in the nodal displacement vector $\{d_e\}$:

$$\begin{cases} N_1(x) = 1 - 3\dfrac{x^2}{l^2} + 2\dfrac{x^3}{l^3},\ N_2(x) = x - 2\dfrac{x^2}{l} + \dfrac{x^3}{l^2} \\ N_3(x) = 3\dfrac{x^2}{l^2} - 2\dfrac{x^3}{l^3},\ N_4(x) = -\dfrac{x^2}{l} + \dfrac{x^3}{l^2} \end{cases}. \tag{13}$$

As the polynomial (12) represents an approximate solution of the governing Equations (3), it result the residuum (error or discrepancy):

$$\varepsilon(x) = EI\frac{\mathrm{d}^4 w_e(x)}{d\,x^4} - \bar{k}_1(x)\frac{\mathrm{d}^2 w_e(x)}{\mathrm{d}x^2} + k(x)w_e(x) - q(x) \neq 0. \tag{14}$$

The minimizing of this residuum means to the annulment of Galerkin balanced functional where the weight is considered for each of the four functions, $N_i(x)$. So the element stiffness matrix resulting are [6]…[11]:

$$[k_e] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}; \tag{15}$$

$$[k_w] = \\ = \frac{l}{840}\begin{bmatrix} 24\left(10k_{(1)} + 3k_{(2)}\right) & 2l\left(15k_{(1)} + 7k_{(2)}\right) & 54\left(k_{(1)} + k_{(2)}\right) & -2l\left(7k_{(1)} + 6k_{(2)}\right) \\ 2l\left(15k_{(1)} + 7k_{(2)}\right) & l^2\left(5k_{(1)} + 3k_{(2)}\right) & 2l\left(6k_{(1)} + 7k_{(2)}\right) & -3l^2\left(k_{(1)} + k_{(2)}\right) \\ 54\left(k_{(1)} + k_{(2)}\right) & 2l\left(6k_{(1)} + 7k_{(2)}\right) & 24\left(3k_{(1)} + 10k_{(2)}\right) & -2l\left(7k_{(1)} + 15k_{(2)}\right) \\ -2l\left(7k_{(1)} + 6k_{(2)}\right) & -3l^2\left(k_{(1)} + k_{(2)}\right) & -2l\left(7k_{(1)} + 15k_{(2)}\right) & l^2\left(3k_{(1)} + 5k_{(2)}\right) \end{bmatrix}; \tag{16}$$

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$$[k_t]=\frac{1}{30l}\begin{bmatrix} 3\left(11\bar{k}_{1(1)}+\bar{k}_{1(2)}\right) & -3l\left(\bar{k}_{1(1)}-2\bar{k}_{1(2)}\right) & -3\left(11\bar{k}_{1(1)}+\bar{k}_{1(2)}\right) & 3l\left(2\bar{k}_{1(1)}-\bar{k}_{1(2)}\right) \\ 3l\bar{k}_{1(1)} & l^2\left(3\bar{k}_{1(1)}+\bar{k}_{1(2)}\right) & -3l\bar{k}_{1(1)} & -l^2\bar{k}_{1(2)} \\ -3\left(\bar{k}_{1(1)}+11\bar{k}_{1(2)}\right) & 3l\left(\bar{k}_{1(1)}-2\bar{k}_{1(2)}\right) & 3\left(\bar{k}_{1(1)}+11\bar{k}_{1(2)}\right) & -3l\left(2\bar{k}_{1(1)}-\bar{k}_{1(2)}\right) \\ 3l\bar{k}_{1(2)} & -l^2\bar{k}_{1(1)} & -3l\bar{k}_{1(2)} & l^2\left(\bar{k}_{1(1)}+3\bar{k}_{1(2)}\right) \end{bmatrix}, \tag{17}$$

were,

$$\left(\ [k_e]+[k_w]+[k_t]\ \right)\{d_e\}=\{S_e\}-\{R_e\} \tag{18}$$

is the matrix notation of the governing differential equation (3) and $[k_e]$ is the stiffness matrix of the flexure beam element; $[k_w]$ is the stiffness matrix of the first subgrade parameter (springs layer); $[k_t]$ is the stiffness matrix of the second subgrade parameter.

The vector $\{R_e\}$ depends on the distributed load on the element and, for $q(x) = q =$ *const*., it result

$$\{R_e\}=\left\{\frac{ql}{2}\ \frac{ql^2}{12}\ \frac{ql}{2}\ -\frac{ql^2}{12}\right\}^T. \tag{19}$$

4. PROGRAM GENERAL DESCRIPTION

A typical finite element program consists of the following sections [12] (Figure 5):

1. Preprocessing section.
2. Processing section.
3. Post-processing section.

In the preprocessing section the data and structures that describe the particular problem statement are defined. These include the finite element discretization, material properties, solution parameters, etc.

The processing section is where the finite element objects (*e.g.* stiffness matrices, force vectors, etc.) are computed, boundary conditions are enforced and the system is solved.

The post-processing section is where the results from the processing section are analyzed. Here stresses may be calculated and data might be visualized.

The Graphical User Interface (GUI) of the EBBEF2p lets the user to interactively enter the operations from the above sections. Although many pre and post-processing operations are already programmed in MATLAB, the EBBEF2p environment have build-functions that come to answer to a particular problem

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statement. The procedure used to obtain a complet solution for a beam resting on two-parameter foundation is indicated below.

A EBBEF2p work session is started with calling the main function file *input.m*, from the MATLAB prompt. The functions that are integrated here (*draw*, *write* and user interface controls – *mnu* function) aids the user in generating data defining the finite element model, for a given problem, with taking advantage of a fully functional GUI (Figure 6).

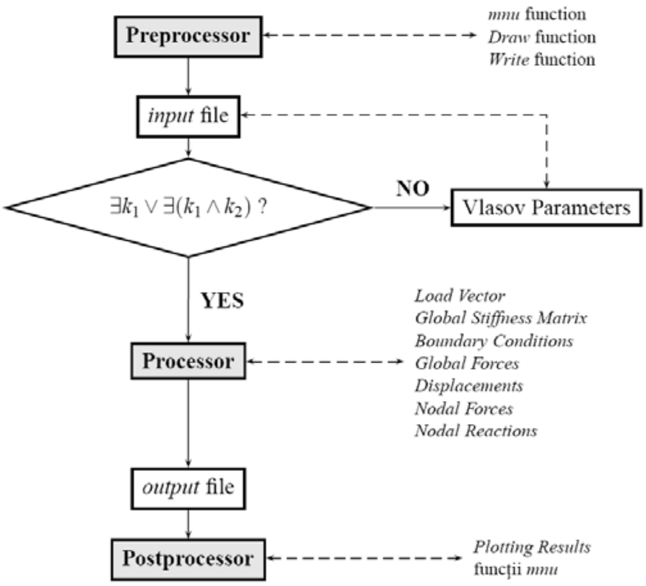

Figure 5. Flow chart for EBBEF2p.

All the finite element data (geometry, material properties, loads, supports, soil parameters and generated FE mesh) are write in a binary data file *input.mat*.

Before EBBEF2p solver (processor) initialization (*ebbef2p.m* function file) are checked the subgrade conditions:

- if exist k_s or (k_s and k_1) values, the finite element model is analysed with these one;
- in this case, the user is required to put in distributions of the deformation modulus E_s, the Poisson number ν and a depth of influence zone H along a beam (if these are not introduced apriory). The program then exploits these parameters to compute values k_s and k_1 using the iterative procedure described foregoing.

In the locations where there is no subsoil, the user can simply set to zero k_s and k_1 parameters (or E_s).

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By collecting element data from the input file, development of the elements stiffness matrices, which are assembling into the global stiffness matrix by using the direct stiffness approach, is done. Partitioning the global stiffness matrix by applying boundary condition (forces, displacements, supports), the remain matrix equation is solved by using Gaussian elimination. With the obtained solution, the displacements, global and nodal forces are calculated and saved to the binary data file *output.mat*.

Finaly, the data stored in *output* file are visualized by the help of MATLAB build-in plot function.

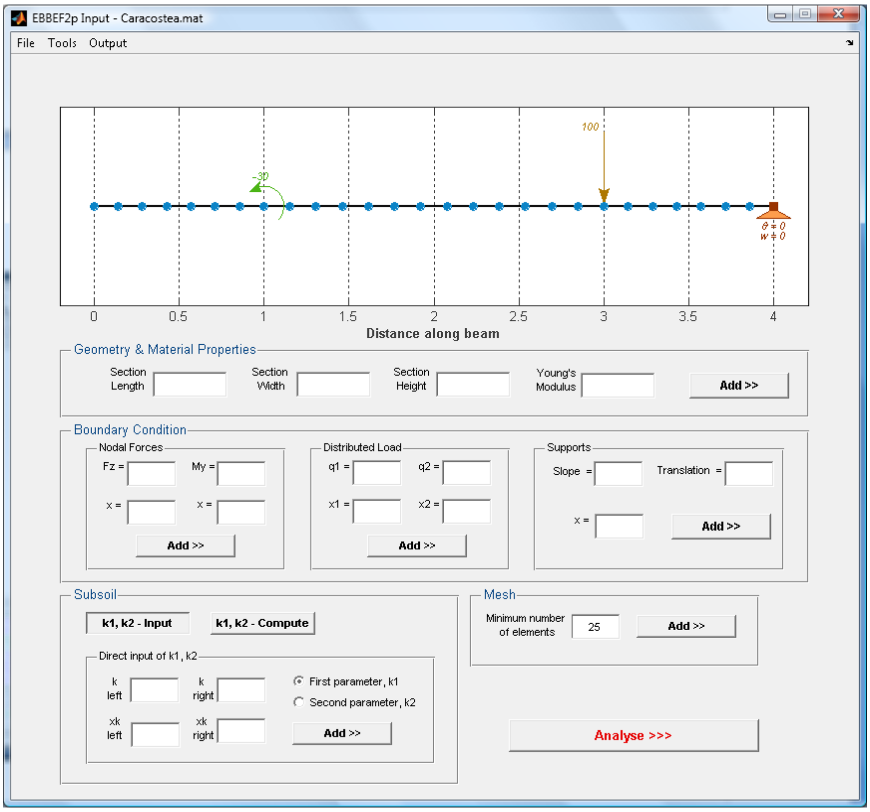

Figure 6. Sample input screen of EBBEF2p.

5. EXAMPLE PROBLEMS. BENCHMARKING

In order to give a comparison between the EBBEF2p solution and the other solution from theory of beams on elastic foundation, a few example problems are presented below.

5.1. Beam on Winkler Foundation

With the general footing and load data shown in Figure 7 (after [13]), the correponding EBBEF2p model is given in Figure 8.

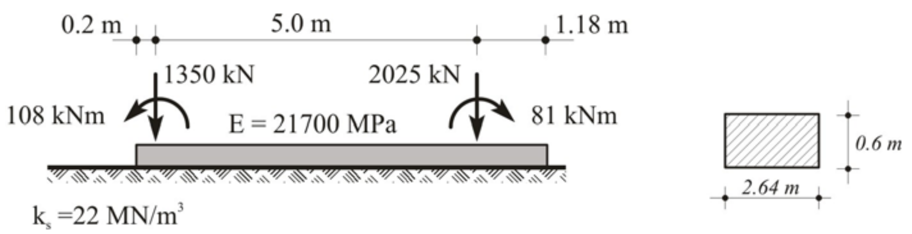

Figure 7. Geometry and load data for the considered example.

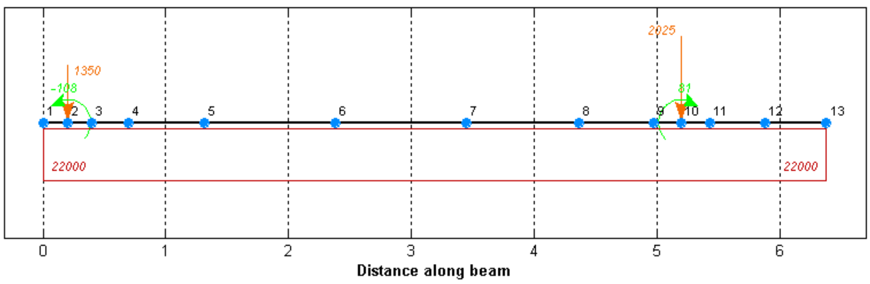

Figure 8. EBBEF2p finite element model.

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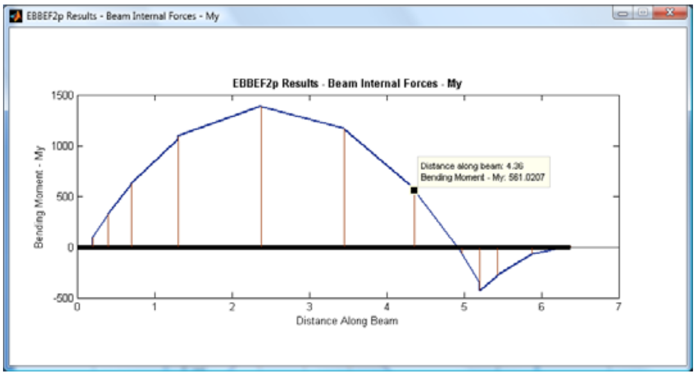

Figure 9. Sample EBBEFp output screen for bending moment.

The results of the numerical analyses have been summarized in Table 1 (maximum values) and a typical EBBEFp output for bending moment pattern is shown in Figure 9.

Table 1. Comparison between EBBEF2p and Bowles solution.

	EBBEF2p	Bowles [13]
Rotation (at node 1), [rad]	0.00269	0.00253
Vertical displacement (at node 1), [mm]	12.3	11.8
Bending moment (at node 6), [kNm]	1387.1	1223.3
Shearing force (at node 10, left), [kN]	1236.7	1190.7
Soil pressure (at node 1), [kN/m]	271.3	260.1

5.2. Fixed-End Beam on Winkler Foundation

The example problem shown in Figure 10 is solved in [14] by the Umanski's method, only for the bending moment.

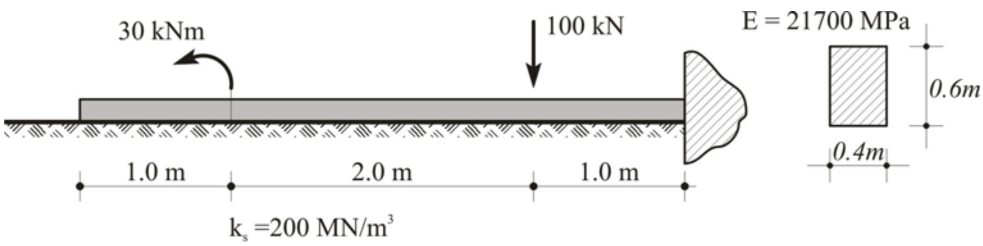

Figure 10. Fixed-end beam on Winkler foundation.

A total of 27 two-node beam element were used to define the corresponding EBBEF2p finite element mesh shown in Figure 11.

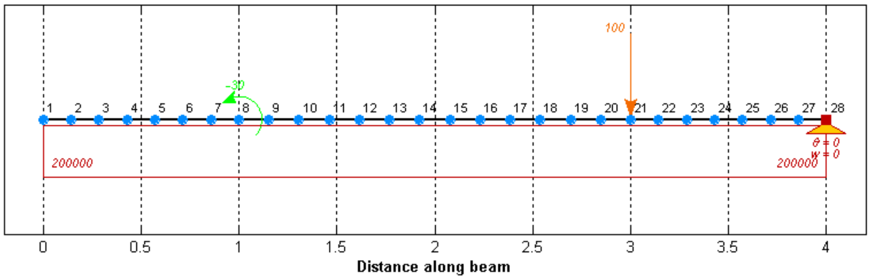

Figure 11. EBBEF2p finite element model.

The computed bending moment pattern is shown in Figure 12. It can be noted that the obtained solution is fairly close to those by Umanski's method.

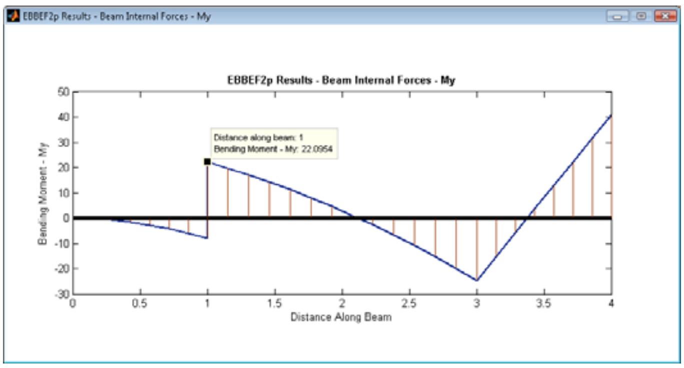

Figure 12. EBBEFp output for bending moment.

5.3. Beam on Modified Vlasov Foundation

As numerical example, a beam of length L = 20 m, width b = 0.5 m and height h = 1.0 m with modulus of elasticity E = 27000 MPa is considered to be suported by foundation having depth H = 5 m, deformation modulus E_s = 20 MPa and Poissons ratio ν_s = 0.25 (Figure 13). A concentrated load P = 500 kN applied at the center of the beam, is assumed. The results of this example problem are compared with solutions obtained by two-dimensional finite element plane strain analyses (2D FEM).

A total of 905, 15-noded triangular elements with a fourth order interpolation for displacements and twelve Gauss points for the numerical integration were used to define the mesh for the 2D FEM model.

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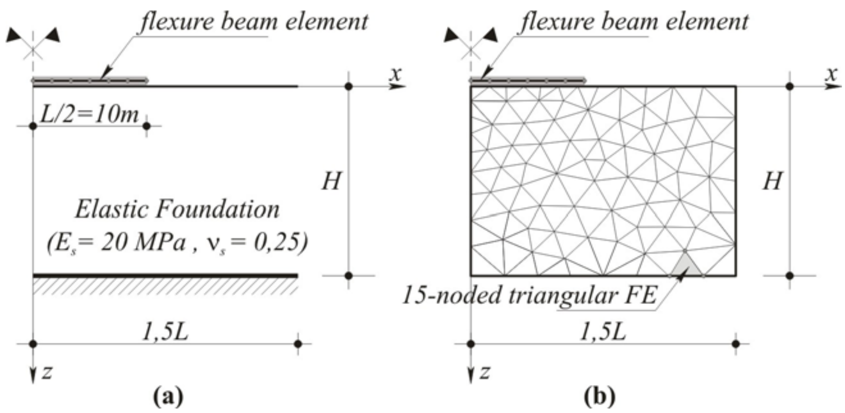

Figure 13. Geometry of the considered example: (**a**) – EBBEF2p; (**b**) – 2D FEM.

In both EBBEF2p and 2D FEM models, the beam is modeled with flexure beam element (Table 2).

Table 2. Beam modelling.

	EBBEF2p model	2D FEM model
Element type	linear with 2 nodes	linear with 5 nodes
Total number of nodes	35	469
Total number of elements	34	117

The results from both 2D FEM and EBBEF2p technique are shown for comparison in Figure 14. It can be noted that both solution have almost the same shape and they are in good agreement. However, a full comparison between these two technique is not fair, because in the 2D finite element solution, complete compatibility of displacements at the beam-soil interface is assumed, but only vertical displacement compatibility exists in Vlasov model [5].

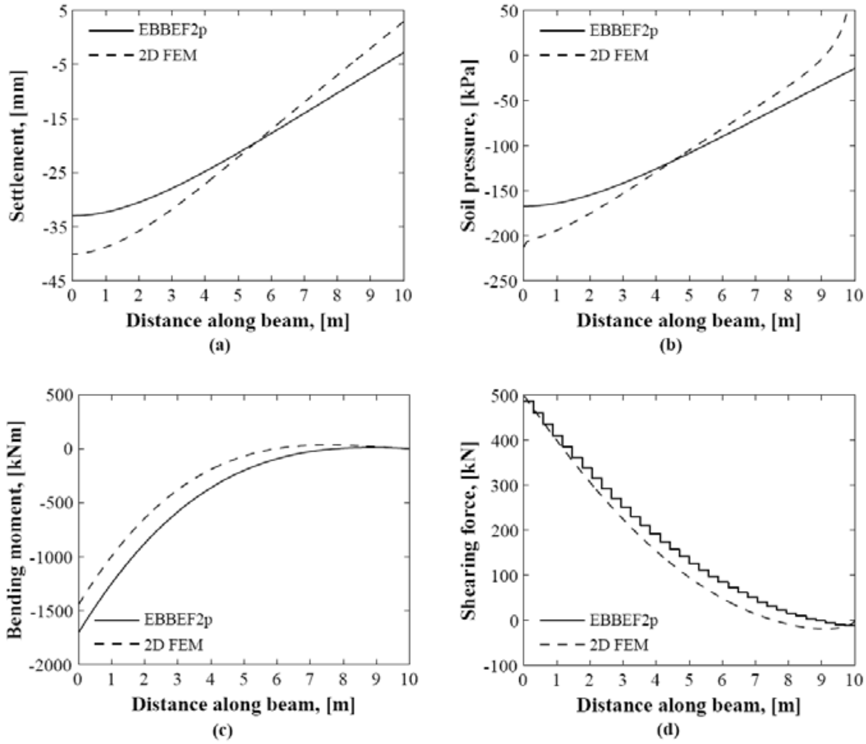

Figure 14. EBBEF2p *vs.* 2D FEM solution: (**a**) – settlement; (**b**) – soil pressure; (**c**) – bending moment; (**d**) – shearing force.

The results of the final computed values of the soil parameters are presented in Table 3. This demonstrate the versatility of the modified Vlasov foundation model, programmed in EBBEF2p: solve beam on elastic foundation problems without having a need to establish the values of foundation parameters.

Table 3. The obtained values of Vlasov foundation parameters.

k, [kN/m^2]	$\overline{k}_1$, [kN]	γ	Number of iterations
2401.57	6515.22	0.418	3

6. CONCLUSIONS

A computer program called EBBEF2p has been developed in MATLAB environment in order to performs complete static structural analysis of beams which rests on one or two-parameter elastic foundation for any loading and boundary condition. By considering a linear variation of both foundation parameter, EBBEF2p can account in a consistent way for the bearing soil inhomogeneity.

The performance and accuracy of EBBEF2p has been carefully tested by carrying out analyses of problems with known solution or comparing results with solutions obtained on numerical model more complex. As a general observation, the obtained EBBEF2p solution are reasonably close to those from theory of beams on elastic foundation and also in good agreement with more sophisticated finite element solutions.

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