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A new look into finite element templates

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Summary

The present paper is dedicated to evaluation and new method of construction the finite element templates. A template is an algebraic form of element matrices, which contains free parameters. Setting the parameters to specific values produces element instances. Two templates are analyzed: Bernoulli and Timoshenko beam. The number of free parameters is discussed by a general method.

KEYWORDS: finite element template, consistency, ellipticity.

1. INTRODUCTION

Beams, plates and shells are widely considered in engineering applications. However the corresponding discretization procedures are not yet sufficiently reliable. It is difficult to obtain an element that is optimal. In a formulation we should aim to satisfy: ellipticity, consistency and the inf-sup condition.

Ellipticity ensures that the finite element model is solvable and physically means there are no spurious zero energy modes. This condition can easily be verified by studying the zero eigenvalues and corresponding eigenvectors of the stiffness matrix of a single unsupported finite element.

Consistency is related to the convergence. The finite element solution must converge to the solution of a mathematical problem the element size h is close to zero. The bilinear forms used in the finite element discretization must approach the exact bilinear forms of the mathematical model as h approaches zero. The *inf-sup* condition ensures optimal convergence in bending-dominated problems and is not a subject of this paper.

One of the interesting concepts are finite element templates proposed by Felippa [1,2]. The objective of this paper is to evaluate finite element templates, proposed by Felippa, with the use of the energy-difference criterion (to check if the template is consistent) and the spectral analysis (to check if the template is elliptical). Two beam templates are discussed: for Bernoulli theory and for Timoshenko theory. A new, general method for development of finite element templates is proposed.

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2. TEMPLATES

A finite element template is an algebraic form for element matrices, which contains free parameters (Felippa [1,2]). Setting those parameters to specific values produces element instances. The set of free parameters is called the template *signature.* Borrowing the terminology from biogenetics, the signature may be viewed as an “element DNA” that uniquely characterizes it as an individual entity. Elements derived by different techniques thet share the same signature are called *clones.* The template should fulfill the following conditions:

- consistency (the individual element test is passed),
- stability (correct rank and nonnegativity conditions),
- parametrization (free parameters)
- invariance (the element is observer invariant).

The element stiffness matrix derived through the template approach is based on the fundamental decomposition

$$\mathbf{K} = \mathbf{K}_b(\alpha_i) + \mathbf{K}_h(\beta_j)\,,$$

where $\mathbf{K}_b$ and $\mathbf{K}_h$ are the basis and higher-order stiffness matrices respectively.

α_i, β_j are free parameters. These two matrices play different and complementary roles. The basic stiffness $\mathbf{K}_b$ takes care of consistency and element-type-mixing. The higher order stiffness $\mathbf{K}_h$ is a stabilization term that provides the correct rank and may be adjusted for accuracy.

3. BERNOULLI BEAM TEMPLATE

Let us consider a finite element template for Bernoulli beam [1] of two-noded element of the length L with natural d.o.f. $\mathbf{q}_e = \{w_i, \phi_i, w_k, \phi_k\}$.

$$\mathbf{K}^{Template}_{Bernoulli} = \mathbf{K_b} + \mathbf{K_h} = \frac{EJ}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + \beta\frac{EJ}{L^3}\begin{bmatrix} 4 & 2L & -4 & 2L \\ 2L & L^2 & -2L & L^2 \\ -4 & -2L & 4 & -2L \\ 2L & L^2 & -2L & L^2 \end{bmatrix}$$

For $\beta = 3$ the well known beam element with Herimite’s polynomial shape functions. The template depends on one free parameter β. The energy-difference procedure (described in details in [3,5], and used for evaluation of beam and plate finite elements in [4,5]) can be used to check if the template satisfies the consistency condition.

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The strain energy density of a beam is expected to be

$$2\tilde{E}_s = EJ\left(\frac{d^2 w}{dx^2}\right)^2 .$$

The equivalent density of element strain energy is a quadratic form

$$2\tilde{E}_s^{ES-Template} = \frac{1}{L}\mathbf{q}_e^T \mathbf{K}_{Bernoulli}^{Template} \mathbf{q}_e .$$

The nodal displacements can be expressed by average displacements and its derivatives in the midpoint of the element with the use of Taylor series expansion

$$w_i = w(x) - \frac{\Delta w}{\Delta x}(x)\frac{L}{2} + \frac{1}{2}\frac{\Delta^2 w}{(\Delta x)^2}(x)\left(\frac{L}{2}\right)^2 - \frac{1}{6}\frac{\Delta^3 w}{(\Delta x)^3}(x)\left(\frac{L}{2}\right)^3 + \ldots,$$

$$w_k = w(x) + \frac{\Delta w}{\Delta x}(x)\frac{L}{2} + \frac{1}{2}\frac{\Delta^2 w}{(\Delta x)^2}(x)\left(\frac{L}{2}\right)^2 + \frac{1}{6}\frac{\Delta^3 w}{(\Delta x)^3}(x)\left(\frac{L}{2}\right)^3 + \ldots,$$

$$\phi_i = \frac{\Delta w}{\Delta x}(x) - \frac{\Delta^2 w}{(\Delta x)^2}(x)\frac{L}{2} + \frac{1}{2}\frac{\Delta^3 w}{(\Delta x)^3}(x)\left(\frac{L}{2}\right)^2 - \frac{1}{6}\frac{\Delta^4 w}{(\Delta x)^4}(x)\left(\frac{L}{2}\right)^4 + \ldots,$$

$$\phi_k = \frac{\Delta w}{\Delta x}(x) - \frac{\Delta^2 w}{(\Delta x)^2}(x)\frac{L}{2} + \frac{1}{2}\frac{\Delta^3 w}{(\Delta x)^3}(x)\left(\frac{L}{2}\right)^2 - \frac{1}{6}\frac{\Delta^4 w}{(\Delta x)^4}(x)\left(\frac{L}{2}\right)^4 + \ldots .$$

After collecting the expressions with respect to L we have

$$2\tilde{E}_s^{ES-Template} = EJ\left(\frac{\Delta^2 w}{\Delta x^2}\right)^2 + L^2 EJ\left[\frac{\beta}{36}\left(\frac{\Delta^3 w}{(\Delta x)^3}\right)^2 + \frac{1}{12}\frac{\Delta^2 w}{\Delta x^2}\frac{\Delta^4 w}{\Delta x^4}\right] + 0(L^4).$$

In the limit case $L \to 0$ the following relation is valid

$$\lim_{L\to 0}\tilde{E}_s^{ES-Template} = \tilde{E}_s .$$

This is a proof that the element template satisfies the consistency requirement for any β. The basis stiffness matrix $\mathbf{K}_b$ is responsible for the first term of the strain energy. Formally the β parameter is free from zero to infinity.

The element template should also elliptical. To check this condition it is necessary to calculate the eigenvalues and eigenvectors of the template. Two zero eigenvalues related to rigid body motions are expected. The two other eigenvalues

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are to be positive and related to the deformed element. The results are the following

$$\lambda_1 = 0,\ \ \lambda_2 = 0,\ \ \lambda_3 = 2\frac{EJ}{L},\ \ \lambda_4 = 10\beta\frac{EJ}{L},$$

$$\mathbf{w}_1 = \{-L,\ 1,\ 0,\ 1\},$$
$$\mathbf{w}_2 = \{\ 1,\ 0,\ 1,\ 0\},$$
$$\mathbf{w}_3 = \{\ 0,\ -1,\ 0,\ 1\},$$
$$\mathbf{w}_1 = \{\ 2L,\ 1,\ -2L,\ 1\}.$$

It is seen that the element is elliptical if $\beta > 0$. If $\beta \to 0$ the fourth eigenvalue tends to be zero with the eigenvector that describes deformed element.

Since the development of the element template is rather complicated, a couple of questions arises after reading the texts of Felippa [1,2]:

How many free parameters exist in the template?

Do the free parameters exist in both matrices (basis and higher order)? etc.

Let us examine some more general matrices to answer the questions.

$$\mathbf{K}_{Bernoulli}^{Extended} = \frac{EJ}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & -a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & -a_{24} & 0 & a_{44} \end{bmatrix} + \frac{EJ}{L^3}\begin{bmatrix} 4b_{11} & 2Lb_{12} & -4b_{13} & 2Lb_{14} \\ 2Lb_{12} & L^2b_{22} & -2Lb_{23} & L^2b_{24} \\ -4b_{13} & -2Lb_{23} & 4b_{33} & -2Lb_{34} \\ 2Lb_{14} & L^2b_{24} & -2Lb_{34} & L^2b_{144} \end{bmatrix}$$

The basis matrix is responsible for the first term of the strain energy, so it depends only on rotations. There are 3 formally independent parameters in this matrix. The second matrix formally depends on 10 parameters. Following the procedure described above one can receive the following strain energy density of the element:

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$$2\tilde{E}_s^{ES-Extended} = \frac{4}{L^4} EJ(b_{11} - 2b_{13} + b_{33})w^2 +$$
$$+\frac{4}{L^3} EJ(-b_{11} + b_{12} + b_{14} - b_{23} + b_{33} - b_{34})w\frac{\Delta w}{\Delta x} +$$
$$+\frac{1}{L^2} EJ(b_{11} - 2b_{12} + 2b_{13} - 2b_{14} + b_{22} - 2b_{23} + 2b_{24} + b_{33} - 2b_{34} + b_{44})\left(\frac{\Delta w}{\Delta x}\right)^2 +$$
$$+\frac{1}{L^2} EJ(b_{11} - 2b_{12} - 2b_{13} + 2b_{14} + 2b_{23} + b_{33} - 2b_{34})w\frac{\Delta^2 w}{\Delta x^2} +$$
$$+\frac{1}{L^2} EJ(a_{22} - 2a_{24} + a_{44})\left(\frac{\Delta w}{\Delta x}\right)^2 +$$
$$+\frac{1}{2L} EJ(-b_{11} + 3b_{12} - b_{14} - 2b_{22} + b_{23} + b_{33} - 3b_{34} + 2b_{44})\frac{\Delta w}{\Delta x}\frac{\Delta^2 w}{\Delta x^2} +$$
$$+\frac{1}{2L} EJ(-b_{11} + 3b_{12} + 3b_{14} - 3b_{23} + b_{33} - 3b_{34})w\frac{\Delta^3 w}{\Delta x^3} +$$
$$+\frac{1}{L} EJ(-a_{22} + a_{44})\frac{\Delta w}{\Delta x}\frac{\Delta^2 w}{\Delta x^2} +$$
$$+\frac{EJ}{12}(b_{11} - 4b_{12} + 2b_{13} - 4b_{14} + 3b_{22} - 4b_{23} + 6b_{24} + b_{33} - 4b_{34} + 3b_{44})\frac{\Delta w}{\Delta x}\frac{\Delta^3 w}{\Delta x^3} +$$
$$+\frac{EJ}{12}(-b_{12} + b_{23} + b_{14} - b_{34})w\frac{\Delta^4 w}{\Delta x^4} +$$
$$+\frac{EJ}{16}(b_{11} - 4b_{12} - 2b_{13} + 4b_{14} + 4b_{22} + 4b_{23} - 8b_{24} + b_{33} - 4b_{34} + 4b_{44})\left(\frac{\Delta^2 w}{\Delta x^2}\right)^2 +$$
$$+0(L)$$

To fulfill the consistency condition the following equations are to be satisfied:

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_{22} \\ a_{24} \\ a_{44} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}.$$

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$$\begin{bmatrix} 1 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 1 & 2 & -1 \\ 1 & -2 & 1 & 2 & -2 & 1 & -2 & 0 & -2 \\ 1 & -2 & 0 & -2 & 2 & 1 & 2 & 0 & -2 \\ -1 & 3 & -2 & 0 & 1 & 1 & -1 & 0 & -3 \\ -1 & 3 & 0 & 0 & -3 & 1 & 3 & 0 & -3 \\ 1 & -4 & 3 & 2 & -4 & 1 & -4 & 6 & -4 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & -4 & 4 & -2 & 4 & 1 & 4 & -8 & -4 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{22} \\ b_{13} \\ b_{23} \\ b_{33} \\ b_{14} \\ b_{24} \\ b_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ -2 \\ 0 \\ -3 \\ 0 \\ -4 \end{bmatrix} b_{44}$$

There are 3 equations for 3 parameters "*a*". We have

$$a_{22} = a_{24} = a_{44} = 1.$$

It is seen that the basis matrix of Bernoulli beam template is independent on the free parameters.

There are 9 equations for 10 parameters "*b*". Thus

$$b_{44} = \beta,\ b_{11} = b_{12} = b_{22} = b_{13} = b_{23} = b_{33} = b_{14} = b_{24} = b_{34} = b_{44} = \beta.$$

3. TIMOSHENKO BEAM TEMPLATE

Let us consider a two-noded element of the length L with natural d.o.f. $\mathbf{q}_e = \{w_i, \phi_i, w_k, \phi_k\}$. Timoshenko beam template proposed by Felippa [2] is more complex than for Bernoulli beam and depends on 3 parameters:

$$\mathbf{K}_{Bernoulli}^{Template} = \mathbf{K_b} + \mathbf{K_h} =$$

$$= \frac{\alpha EJ}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + 3\beta\frac{EJ}{L^3}\begin{bmatrix} 4 & 2L & -4 & 2L \\ 2L & \psi L^2 & -2L & \psi L^2 \\ -4 & -2L & 4 & -2L \\ 2L & \psi L^2 & -2L & \psi L^2 \end{bmatrix}.$$

Following a similar procedure like for the Bernoulli beam we have:

- Density of strain energy

$$2\widetilde{E}_s = EJ\left(\frac{d\phi}{dx}\right)^2 + H\left(\phi - \frac{dw}{dx}\right)^2$$

- Nodal parameters

$$w_i = w(x) - \frac{\Delta w}{\Delta x}(x)\frac{L}{2} + \frac{1}{2}\frac{\Delta^2 w}{(\Delta x)^2}(x)\left(\frac{L}{2}\right)^2 - \frac{1}{6}\frac{\Delta^3 w}{(\Delta x)^3}(x)\left(\frac{L}{2}\right)^3 + ...,$$

$$w_k = w(x) + \frac{\Delta w}{\Delta x}(x)\frac{L}{2} + \frac{1}{2}\frac{\Delta^2 w}{(\Delta x)^2}(x)\left(\frac{L}{2}\right)^2 + \frac{1}{6}\frac{\Delta^3 w}{(\Delta x)^3}(x)\left(\frac{L}{2}\right)^3 + ...,$$

$$\phi_i = \phi(x) - \frac{\Delta \phi}{\Delta x}(x)\frac{L}{2} + \frac{1}{2}\frac{\Delta^2 \phi}{(\Delta x)^2}(x)\left(\frac{L}{2}\right)^2 - \frac{1}{6}\frac{\Delta^3 \phi}{(\Delta x)^3}(x)\left(\frac{L}{2}\right)^4 + ...,$$

$$\phi_k = \phi(x) + \frac{\Delta \phi}{\Delta x}(x)\frac{L}{2} + \frac{1}{2}\frac{\Delta^2 \phi}{(\Delta x)^2}(x)\left(\frac{L}{2}\right)^2 + \frac{1}{6}\frac{\Delta^3 \phi}{(\Delta x)^3}(x)\left(\frac{L}{2}\right)^4 + ...$$

- Density of the template strain energy

$$2\widetilde{E}_s^{Template} = \alpha EJ\left(\frac{\Delta\phi}{\Delta x}\right)^2 + \frac{12\beta EJ}{L^2}\left(\phi - \frac{\Delta w}{\Delta x}\right)^2 +$$

$$+ \beta EJ3\frac{\Delta^2\phi}{\Delta x^2}\left(\psi\phi - \frac{\Delta w}{\Delta x}\right) - \beta EJ3\frac{\Delta^3 w}{\Delta x_3}\left(\phi - \frac{\Delta w}{\Delta x}\right) + 0(L^2)$$

To fulfill the condition of consistency it is necessary to take $\alpha = 1$ and β in the following form

$$\beta = \frac{HL^2}{12EJ}(1 + ...)$$

with any ψ.

Let us propose more general form of the stiffness decomposition:

$$\mathbf{K}_{\substack{Extended\\Bernoulli}} = \frac{EJ}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & -a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & -a_{24} & 0 & a_{44} \end{bmatrix} + \frac{H}{L}\begin{bmatrix} 4b_{11} & 2Lb_{12} & -4b_{13} & 2Lb_{14} \\ 2Lb_{12} & L^2b_{22} & -2Lb_{23} & L^2b_{24} \\ -4b_{13} & -2Lb_{23} & 4b_{33} & -2Lb_{34} \\ 2Lb_{14} & L^2b_{24} & -2Lb_{34} & L^2b_{144} \end{bmatrix}$$

- Density of the extended template strain energy

$$2\widetilde{E}_s^{Extended} = EJ(a_{22} - 2a_{24} + a_{44})\frac{\phi^2}{L^2} + EJ(-a_{22} + a_{44})\frac{\phi}{L}\frac{\Delta\phi}{\Delta x} +$$
$$+ \frac{EJ}{4}(a_{22} + 2a_{24} + a_{44})\left(\frac{\Delta\phi}{\Delta x}\right)^2 + H(b_{11} - 6b_{13} + 3b_{33})\frac{w^2}{L^2} +$$
$$+ H(3b_{12} - 3b_{23} + 3b_{14} - 3b_{34})\frac{\phi w}{L} + H(-b_{11} + 3b_{33})\frac{w}{L}\frac{\Delta w}{\Delta x} +$$
$$+ \frac{H}{4}(3b_{22} + 6b_{24} + 3b_{44})\phi^2 + H(-3b_{12} + 3b_{23} + 3b_{14} - 3b_{34})w\frac{\Delta\phi}{\Delta x} +$$
$$+ \frac{H}{2}(-3b_{12} - 3b_{23} - 3b_{14} - 3b_{34})\phi\frac{\Delta w}{\Delta x} + \frac{H}{4}(b_{11} + 6b_{13} + 3b_{33}) + 0(L)$$

To fulfill the consistency requirement 3 equations for 3 coefficients "a" and 7 equations for 10 coefficients "b" are to be satisfied:

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} a_{22} \\ a_{24} \\ a_{44} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

This matrix equation is equivalent to the conditions received for the Bernoulli beam template in the previous chapter.

$$\begin{bmatrix} 1 & 0 & 0 & -6 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 & -3 & 0 & 3 \\ -1 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 & 0 & 3 \\ 0 & -3 & 0 & 0 & -3 & 0 & -3 \\ 1 & 0 & 0 & 6 & 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{12} \\ b_{22} \\ b_{13} \\ b_{23} \\ b_{33} \\ b_{34} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ -6 & 0 & -3 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} b_{24} \\ b_{34} \\ b_{44} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

One can receive $a_{22} = a_{24} = a_{44} = 1$.

There are 3 independent constants in the group "b". If we put $b_{24} = C_1$, $b_{34} = C_2$, $b_{44} = C_3$ the other coefficients are the following:

$$b_{11}=1,\ \ b_{12}=-\frac{1}{3}(1+C_2),\ \ b_{22}=\frac{1}{3}(4-6C_1-3C_3),$$
$$b_{13}=\frac{1}{3},\ \ b_{23}=-\frac{1}{3}(1+C_2),\ \ b_{33}=\frac{1}{3},\ \ b_{34}=C_2.$$

The derivation above is a proof that there are 3 independent coefficients in the Timoshenko beam template.

4. CONCLUSIONS

The finite element templates, proposed by Felippa [1,2], are algebraic forms that contains some free parameters. The finite element templates for Bernoulli beam and Timoshenko beam are examined in the paper, from the point of view of the consistency condition. A new way of template construction is proposed. It is confirmed that there is 1 free parameter for Bernoulli beam template and 3 free parameters for Timoshenko beam template. The new method of creation the template can be extended for 2D and 3D problems.

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