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Boundary element method numerical modeling of RC plane stress cracked plate

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Summary

In this paper different methods of distribution solution of RC cracked of the plane stress plates are shown. The differential equation of the cracked plates, using the classical variational method of Lagrange is worked out. The displacements equations with the boundary conditions and compatibility conditions in the crack are obtained. The total differential equations in the class of the two-dimensional general vector functions are shown. In this model the effect of discontinuity general deformation vector is taken into account. As the next the viscoelastic plate model has been derived by the variational method of Gurtin in the space of general function. The numerical results of an approximate method of solutions with boundary element method are shown.

Keywords: boundary element method, viscoelastic plate model, RC plane stress cracked plate

1. INTRODUCTION

The RC concrete plates are non-homogeneous. Therefore the response of so heterogeneous structures and additionally defects caused by cracks in concrete to applied actions is generally nonlinear, due to nonlinear constitutive relationships of the materials, known as mechanical nonlinearity and to second order effects of normal forces, known as geometrical nonlinearity. Regard of defects in form of cracks treated as continuous functions, which are usually based on the continuum mechanics approach, gives unsatisfied solution because of summation of assumption errors and solution errors. Therefore the proper mathematical modeling of plate is so important since the final error appears solely in solution phase.

This paper contains a mathematical model of a reinforced concrete plane stress plate formulated in terms of general functions. The physical hypothesis about discontinuous change of displacement vector, caused by cracking of extension zone in the concrete, was taken in the model. Such assumptions and first investigation for distribution model of RC beam with cracks were made by Borcz in 1963 [1].

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This paper expands distribution beam Borcz's model for RC cracked plane stress plates.

The assumption of distribution theory of Schwartz [2] affords possibilities for precise mathematical description of discontinuity of the plate. Fundamental for understanding of the next consideration are the general distributions of Dirac's-δ with given density on the curve $\Lambda \in R^2$ and following properties:

$$\langle \mathbf{y}\,\delta_\Lambda, \mathbf{j} \rangle = \int_\Lambda \mathbf{y}(\mathbf{x})\mathbf{j}(\mathbf{x})\,d\Lambda \text{, where } \mathbf{x} = (x_1, x_2)\,. \tag{1}$$

$$\left\langle D^\alpha(\chi(\mathbf{x})\delta_\Lambda), \varphi(\mathbf{x}) \right\rangle = (-1)^{|\alpha|} \int_\Lambda \chi(\mathbf{x}) D^\alpha \varphi(\mathbf{x})\,d\Lambda \text{, where } \alpha = (\alpha_1 + \alpha_2)\,,\ |\alpha| = \alpha_1 + \alpha_2\,. \tag{2}$$

Here the functions $\psi(\mathbf{x})$, $\varphi(\mathbf{x})$ and $\chi(\mathbf{x})$ are continuous functions on the curve Λ in the space R^2.

The functionals formulated above for unitary density of function $\psi(\mathbf{x})$ and $\chi(\mathbf{x})$ have the analogue filtering property as for the general distribution of Dirac's-δ. It means that the $\psi(\mathbf{x})$ and $\chi(\mathbf{x})$ have the value of function $\varphi(\mathbf{x})$ or its derivatives respectively for the arguments $\mathbf{x}$ belong to the curve Λ.

The arbitrary plane stress plate is considered. The plate has arbitrary homogeneous boundary conditions and is arbitrary forced, see Figure 1. The region of plate Ω is divided by the curve Λ, means the crack, in two zones Ω_1 and Ω_2 with bound $\partial\Omega_1$ and $\partial\Omega_2$. The curve Λ has two ends Λ_1 and Λ_2. The normal external direction cosines of edge Λ of regions Ω_1 and Ω_2 have different sign. The considered model can be easy generalized to any amount of cracks Λ.

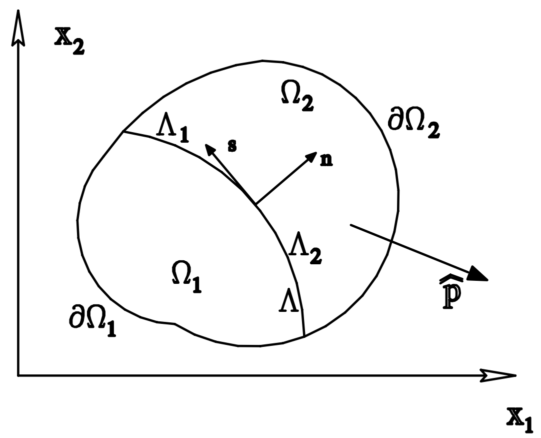

Figure 1. Scheme of plane stress plate with crack Λ

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2. DIFFERENTIAL EQUATION FOR DISPLACEMENT

The discontinuous variational problem of surface integral for displacement was solved. The equilibrium equations, constitutive law and strain equations are assumed to be represented by well-known theory of elasticity relations. We are looking for the extreme of the functional of strain energy U_s with set of permissible displacement value $\mathbf{u}(\mathbf{x})$, where $\mathbf{b}(\mathbf{x})$ means body forces:

$$J[\mathbf{u}(\mathbf{x})] = \int_{\Omega} U_s(\mathbf{u}(\mathbf{x}))d\Omega - \int_{\Omega} \mathbf{b}(\mathbf{x})\mathbf{u}(\mathbf{x})d\Omega - \int_{\partial\Omega} \mathbf{p}(\mathbf{x})\mathbf{u}(\mathbf{x})d\partial\Omega, \tag{3}$$

where $\mathbf{u}(\mathbf{x})$ means displacement vector and $\mathbf{b}(\mathbf{x})$ body forces respectively. The searching function $\mathbf{u}(\mathbf{x})$ is in the class of function $\mathbf{u} \in C^2(\Omega/\Lambda)$ (for $\mathbf{x} \in \Lambda$ function $\mathbf{u}(\mathbf{x})$ has singularity).

Applying Green's transformation with well known relations stress-strain-displacement **S-E-u** we obtain differential equation of plate in plane stress:

$$\mu(\nabla^2 + \frac{3\lambda + 2\mu}{\lambda + 2\mu}\, grad\ div)\ \mathbf{u}(\mathbf{x}) + \mathbf{b}(\mathbf{x}) = 0\ , \tag{4}$$

associated with combination of elementary boundary conditions:

$$\widetilde{\mathbf{P}}(\mathbf{u}(\mathbf{x})) = \mathbf{p}(\mathbf{x})\ , \quad \text{for } \mathbf{x} \in \Omega_1 \cup \Omega_2\ , \tag{5}$$

and compatibility condition in the crack:

$$[\widetilde{\mathbf{P}}(\mathbf{u}(\mathbf{x}))]_{\Lambda} = 0\ , \quad \text{for } \mathbf{x} \in \Lambda_1\Lambda_2\ , \tag{6}$$

where λ and μ are Lame' constants, $\widetilde{\mathbf{P}}$ means operator of surface tension:

$$\widetilde{\mathbf{P}}(.) = \mu(\widetilde{\nabla} + \frac{2\lambda}{\lambda + 2\mu}\ \mathbf{1}\ div)(.)\mathbf{n}\ , \tag{7}$$

where **1** is a unitary tensor, whereas **n** represents normal vector external to the edge $\partial\Omega$.

Here $[\ .\ ]_{\Lambda}$ means difference of left and right side limit of expression in square braces on the curve Λ.

Constitutive law of defect is assumed to be represented as follows (see beam analogy of Borcz [1]):

$$[\mathbf{u}(\mathbf{x})]_{\Lambda_1\Lambda_2} = \mathbf{r}(\mathbf{x})\ , \text{ with condition } \frac{\partial r}{\partial s}(\Lambda_1) = \frac{\partial r}{\partial s}(\Lambda_2) = 0\ . \tag{8}$$

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Here $\mathbf{r}(\mathbf{x})$ means density of defect as a continuous function for $\mathbf{x} \in \Lambda_1\Lambda_2$ and $[\mathbf{u}]_\Lambda = 0$ for $\mathbf{x} \notin \Lambda_1\Lambda_2$. Equation (8) satisfies compatibility condition in the crack, where the displacement vector has a jump on a bound of crack. Here the assumption of internal crack $\Lambda_1\Lambda_2$ was taken. This can be easy proved. Hence on the remaining part of curve Λ the condition $[\mathbf{u}]_\Lambda = 0$ yields, for $\mathbf{x} \notin \Lambda_1\Lambda_2$. Moreover the second condition (8) in the essential way completes the definition of the defect.

It can be easily shown that using Eq. (1)-(2) and Eq. (5)-(8) differential equation (4) has the following form (see [3]).

$$\mu\left\langle (\nabla^2 + \frac{3\lambda + 2\mu}{\lambda + 2\mu}\, grad\, div)\mathbf{u}(\mathbf{x}), \varphi \right\rangle + \left\langle \mathbf{b}(\mathbf{x}), \varphi \right\rangle = \int_{\partial\Omega} [\mathbf{u}(\mathbf{x})\widetilde{\mathbf{P}}(\varphi) + (\mathbf{p}(\mathbf{x}) - \widetilde{\mathbf{P}}(\mathbf{u}(\mathbf{x}))\varphi] d\partial\Omega +$$

$$+ \int_{\Lambda_1\Lambda_2} \mathbf{r}(\mathbf{x})\widetilde{\mathbf{P}}(\varphi) d\Lambda \,. \tag{9}$$

Using functional way of description with distribution in form of Dirac's-δ we can write final general differential equation of RC cracked plate in plane stress, appropriate boundary and compatibility conditions in the crack respectively:

$$\mu(\nabla^2 + \frac{3\lambda + 2\mu}{\lambda + 2\mu}\, grad\, div)\mathbf{u}(\mathbf{x}) = -\widetilde{\mathbf{P}}(\mathbf{r}(\mathbf{x})\delta_\Lambda) + (\mathbf{p}(\mathbf{x}) - \widetilde{\mathbf{P}}(\mathbf{u}(\mathbf{x}))\delta_{\partial\Omega_1} + \widetilde{\mathbf{P}}[(\widetilde{\mathbf{u}}(\mathbf{x}) - \mathbf{u}(\mathbf{x}))\delta_{\partial\Omega_2}] \tag{10}$$

where $\mathbf{b}(\mathbf{x})$ was taken in following form: $\mathbf{b}(\mathbf{x}) = \widetilde{\mathbf{P}}(\widetilde{\mathbf{u}}(\mathbf{x}))\delta_{\partial\Omega_2}$.

The assumption of the jump of displacement vector $\mathbf{u}(\mathbf{x})$ was proved by experimental study [1]. Density of defect also known as constitutive law of crack opening, is the function of tension vector $\mathbf{N}$ acting in the crack:

$$\mathbf{r}(\mathbf{x}) = \mathbf{r}^0(\mathbf{x}) - \mathbf{r}^1(\mathbf{x})\Im(\mathbf{N}(\mathbf{x}))\big|_\Lambda \quad , \quad \mathbf{x} \in \Lambda_1\Lambda_2 \quad . \tag{11}$$

Here $\mathbf{r}^0$ describes residual general deformations whereas $\mathbf{r}^1\Im(\mathbf{N})$ elastic general deformations respectively.

The zone of plate Ω_1 is connected with another one Ω_2 by means of reinforcement bars appearing in the cracks. So, the edges of the cracks are not free from tensions at the points of connections.

The components $\mathbf{r}$ of Eq. (11) are given from RC element tests as well as from general assumption of crack theory and equilibrium conditions in the crack.

Using definition of convolution and Green function satisfying equation $\Delta\Delta\mathbf{G}(\mathbf{x}) = \delta(\mathbf{x})$ the solution of Eq. (10) is described in following form:

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$$\mathbf{u}(\mathbf{x}) = \int_{\Lambda_1\Lambda_2} [\mathbf{r}^0(\mathbf{y}) + \mathbf{r}^1(\mathbf{y})\Im(\mathbf{N})]\widetilde{\mathbf{P}}(\mathbf{G}(\mathbf{x},\mathbf{y}))d\Lambda +$$

$$+\int_{\partial\Omega} \{\widetilde{\mathbf{P}}(\mathbf{G}(\mathbf{x},\mathbf{y}))[\mathbf{u}(\mathbf{y}) - \widetilde{\mathbf{u}}(\mathbf{y})] - \mathbf{G}(\mathbf{x},\mathbf{y})[\widetilde{\mathbf{P}}(\mathbf{u}(\mathbf{y})) - \mathbf{p}(\mathbf{y})]\}d\partial\Omega \; . \qquad (12)$$

Here the curvilinear integrals for the edge Λ can be interpreted as a some external force modeling the defect. It can be proved [3] that for plane stress these integrals are the normal and tangential dipole forces cause the jump of displacement vector in the crack. The acting forces are self-balanced and do not cause the increments of external loading of construction.

The final solution of singular integro-differential Eq. (12) describes an accurate mathematical model of cracked RC in plane stress.

The presented solution includes the discontinuity of general deformations in the crack places and simultaneously satisfying continuity of general tension vector on both sides of the defect Λ. Such formulated model of RC cracked plate will be solved with the help of boundary element method (BEM).

3. DIFFERENTIAL EQUATION OF VISCOELASTIC PLATE

The same way as in chapter 2 was applied to describe solution of viscoelastic plane stress plate. The equilibrium equations in form: $div\ \mathbf{S} + \mathbf{b} = \mathbf{0}$, and geometrical relations in form: $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) = \widetilde{\nabla}\mathbf{u}$.

The set of field equations is fulfilled in the space $\Omega \times [0,\infty)$, where $[0,\infty)$ is time interval. The initial condition of strains' tensor has to be added $\mathbf{E}(\ .\ ,0) = \mathbf{E}^0$, for $t = 0$.

The physical law well known as Boltzmann type is taken in following form:

$$\mathbf{1}*\mathbf{S} = \psi_1 * \mathbf{E} + \frac{1}{2}(\psi_2 - \psi_1)*\mathbf{1}\ tr\ \mathbf{E} - \mathbf{F} \;, \quad \text{where:} \qquad (13)$$

$$\mathbf{F} = \psi_1 * \mathbf{E}^0 + \frac{1}{2}(\psi_2 - \psi_1)*\mathbf{1}\ tr\ \mathbf{E}^0 \;, \qquad (14)$$

with convolution rule: $f(t)*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$. Here the functions ψ_1 and ψ_2 are the functions of relaxation.

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The discontinuous viscoelastic variational problem of Gurtin type, in the some way as considered in chapter 2, was solved. The equilibrium equations, Boltzmann constitutive law, strain equations and initial condition are assumed to be represented by well-known theory of elasticity relations.

Analogue to the functional (3) we are looking for the extreme of the functional of strain energy U_v with set of permissible displacement value $\mathbf{u}(\mathbf{x})$:

$$J[\mathbf{u}(\mathbf{x},t)] = \int_{\Omega} U_v(\mathbf{u}(\mathbf{x},t))d\Omega - \int_{\Omega} \mathbf{1} * \mathbf{b}(\mathbf{x},t) * \mathbf{u}(\mathbf{x},t)d\Omega - \int_{\partial\Omega} \mathbf{1} * \tilde{\mathbf{p}}(\mathbf{x},t) * \mathbf{u}(\mathbf{x},t)d\partial\Omega \,, \quad (15)$$

Applying Green's transformation with material and field relations we obtain differential equation of plate in plane stress:

$$\mathbf{1} * \mathbf{S} = \psi_1 * \mathbf{E} + \frac{1}{2}(\psi_2 - \psi_2) * \mathbf{1}\ tr\ \mathbf{E} - \mathbf{F} \,, \quad (16)$$

associated with combination of elementary boundary conditions for free and fixed edges respectively:

$$[\tilde{\mathbf{P}}(\mathbf{u}(\mathbf{x})) - \mathbf{1} * \tilde{\mathbf{p}}(\mathbf{x})]\big|_{\partial\Omega_1} = 0 \quad \vee \quad \delta \mathbf{u}(\mathbf{x})\big|_{\partial\Omega_2} \,, \quad (17)$$

and compatibility condition in the crack:

$$[\tilde{\mathbf{N}}(\mathbf{u}(\mathbf{x},t))]_{\Lambda} = 0 \,, \qquad \text{for } \mathbf{x} \in \Lambda_1 \Lambda_2 \,, \quad (18)$$

where $\tilde{\mathbf{N}}$ means viscoelastic operator as the analogy of surface tension from theory of elasticity:

$$\tilde{\mathbf{P}}(.) = \mu(\tilde{\nabla} + \frac{2\lambda}{\lambda + 2\mu}\ \mathbf{1}\ div)(.)\mathbf{n} \,, \quad (19)$$

Taking constitutive law of cracks (8) and applying functional way of description with δ of Dirac's type, the final general differential equation of viscoelastic RC cracked plate in plane stress, appropriate boundary, compatibility and initial conditions respectively, can be written as follows:

$$[\psi_1 * \nabla^2 + \frac{1}{2}(\psi_1 + \psi_2) * grad\ div]\mathbf{u}(\mathbf{x},t) + \mathbf{1} * \mathbf{b}(\mathbf{x},t) - div\ \mathbf{F} =$$

$$= -\tilde{\mathbf{N}}(\mathbf{r}(\mathbf{x},t)\delta_{\Lambda}) + [\mathbf{1} * \tilde{\mathbf{p}}(\mathbf{x},t) - \tilde{\mathbf{N}}(\mathbf{u}(\mathbf{x},t))]\delta_{\partial\Omega_1} + \tilde{\mathbf{N}}[(\tilde{\mathbf{u}}(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t))]\delta_{\partial\Omega_2} \,. \quad (20)$$

Here the constitutive law of crack opening is expanded as a rule valid additionally in time. Note, that the final solution (20) is similar to elastic solution (10), difference occurs only for relaxation function with λ and μ as a time depended function.

The solution of Eq. (20) is possible with help of elastic solution as a first approximation of viscoelastic solution. It denotes the solution of "associated" elastic problem $\mathbf{u}(\mathbf{x},t)$ from Eq. (12) can be used to convolutions' solution of viscoelastic static problem of RC cracked plain stress as follows:

$$\mathbf{u}(\mathbf{x},t)=\int_0^t \frac{\partial \mathbf{u}(\mathbf{x},\tau)}{\partial \tau}\,\varphi(t-\tau)d\tau \;, \qquad (21)$$

where φ is the function with the combination of relaxation and creep functions.

4. MODELING BY BOUNDARY ELEMENT METHOD

Deformation behavior depends on the history of the loading as well as nonlinearity of material properties. Hence, equations and definitions of the boundary element method in rate form according to the Brebbia [4] formulations were assumed. According to small strains' theory, total strain rate for inelastic problem can be divided into an elastic and inelastic part of total strain rate tensor respectively. Herein, the inelastic strains mean any kinds of strain field that can be considered as initial strains, that is, plastic or viscoplastic strain rate, creep strain rate, thermal strain rate and strain rate due to other causes. So now, we can write (see also [3]) the equations of considered problem with non-linear BEM formulations for fictitious traction vector **p** and body forces **b**, finally leading to initial stresse $\boldsymbol{\sigma}^p$:

$$\mathbf{H}\dot{\mathbf{u}}-\mathbf{A}\dot{\mathbf{p}}=\mathbf{B}\dot{\sigma}^p+\mathbf{F}\dot{\mathbf{x}}+\dot{\mathbf{Q}}\;(\mathbf{x}) \;, \qquad (22)$$

where: **u** - displacement vector,
X - vector of unknown edge traction,
p - vector of fictitious traction,
$\boldsymbol{\sigma}^p$ - vector of initial stresses,
H, **A** - the same matrices as for elastic analysis,
B - matrix due to the inelastic stress integral,
F - matrix refers to the fundamental function cause by forcing traction with vector **X**, that is, modeling density of crack opening for plane stress ,
Q - matrix of bond, bond-slip relations and other displacements due to aggregate interlock and dowel action of reinforcement in the crack, related to displacement **u**.

5. THE INCREMENTAL COMPUTATIONS

The results' correctness depends on the choice of right type boundary or finite elements respectively and a careful discretization of the structure. The influence of the above on the problem to be studied cannot be neglected. The appropriate simulation of the load-carrying behavior of RC structure is more important then the accuracy of the numerical calculations. The question what kind of numerical methods, boundary or finite elements should be preferably chosen cannot be answered satisfactorily. Equations (22) must be solved numerically with iterative and incremental techniques. Iteration results due the fact that the right sides of equations depend directly on searching functions. In addition searching functions depend indirectly on physical law. The incremental computation is caused by rate form of Eq. (22). For iteration and incremental computations the modified Newton-Raphson method was applied.

For the computation of plate the behavior of concrete is taken as well known stress-strain Madrid parabola. The stress-strain relation of steel bars was taken as a well-known elasto-plastic relation from uniaxial tests.

The creep of concrete was taken from the Bažant's [5] rheological model. This model is most suitable for concrete structures because the parameters can be calculated only from the concrete composition (for basic and drying creep as well as shrinkage). The creep function of Bažant is shown below (f_c' means the 28 days compressive strength of concrete):

$$J(t,\tau)=\frac{1}{E_o(f_c')}[1+\varphi_1(f_c')(\tau^{-m(f_c')}+\alpha)(t-\tau)^{n(f_c')}] \tag{23}$$

After cracking of concrete, the tensile forces in the cracked area are transmitted by bond to the reinforcement consisting of steel bars. Along the segments of broken adhesion the steel bar co-operates with the concrete trough the tangential stresses distributed on the perimeter of the bar. The slip is defined as a relative displacement between reinforcement bars and surrounding concrete. The increment of tensile stresses in the steel bar was approximated by the third-degree curve. Hence, the tangential stresses and bond-slip relationships, as representation of the stiffness of the bond has been found to be in the agreement to tests of Dörr [6], (that is, the second-degree distribution along the segment l_f , where l_f means distance between cracks).

The time-dependence of bond in the loaded state exhibits a similar behavior as concrete in compression (see [7]). The presupposition similar to the linear creep theory of concrete in compression is used for bond creep with bond creep coefficient φ_b. Naturally, in accordance to τ-Δ relationship of Dörr [6], bond creep

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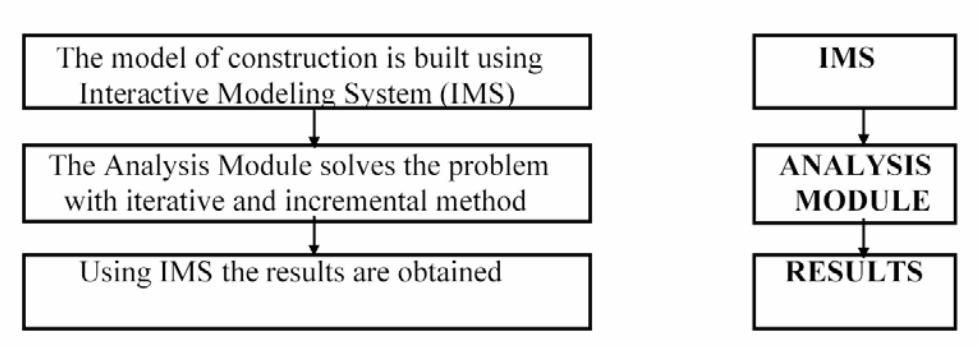
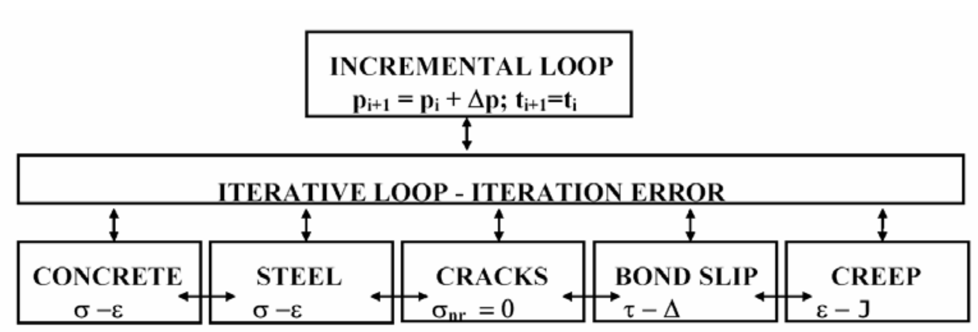
cannot be described with linear theory. The model of Rotasy [7] was applied to describe the creep of the bond in the cracked concrete.

The development of "rotating cracks" is considered as single cracks treated as the boundary element where the direction of the crack has to be assumed in accordance with the previous step of the main direction of the tensile stresses.

A program for Boundary Element Analysis named PLATE for planar structure was designed. The route through a PLATE analysis includes following procedures is shown below:



The way of applying iterative and incremental technique in the Analysis Module is shown below in the block diagram:



6. NUMERICAL EXAMPLE

The results of simply supported square panel WT3, tested by Leonhardt and Walther [8], were taken to check BEM solution for plain stresses. The panel was reinforced horizontally in different way for bottom and top part. The bottom zone (Ra_1) had 2ϕ8 mm bars each 6 cm fixed in 4 rows, the top zone and vertical bars (Ra_2) were 2ϕ5 mm each 26 cm.

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In Figure 2 the experiment received load-midspan deflection relation of the panel as compared to the results of BEM and other authors ([9] and [10]) FEM numerical calculations.

Figure 3 shows the dependence of the time and loading levels on the crack width a_f for panel WT3.

7. CONCLUSIONS

The numerical results obtained for the problems of panels indicated that the presented methods are capable to predict sufficiently and satisfactorily response of reinforced concrete planar structure.

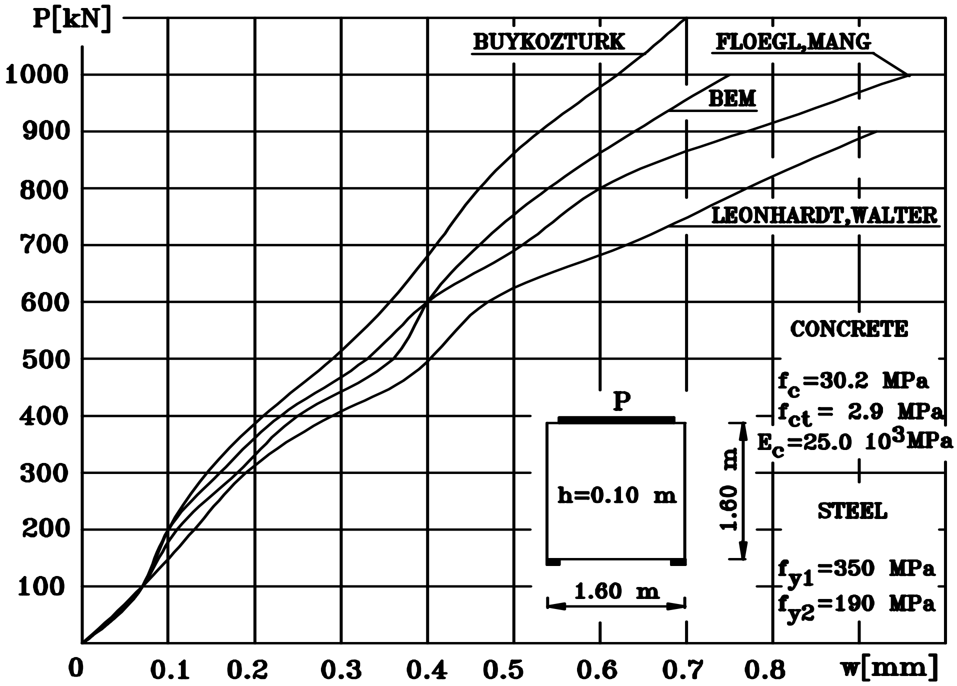

Figure 2. Comparison of calculated load-midspan deflection relations with the test result of panel WT3 [8]

For the computation of planar structure the behavior of concrete should be considered in the biaxial domain. The concrete properties are influenced by many different factors. Therefore the biaxial stress-strain relation and the failure criterion of concrete depend on the results of the tests that are performed to obtain these relations. The biaxial tests of Kupfer [11] for short time loading and proportionally increasing load proved to be the most reliable. Different authors have used these

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test results to develop analytical formulation of the failure and deformation behavior of the concrete. Link [12] developed an incremental formulation for the tangent stiffness of the concrete on the basis of Kupfer's tests. The stresses are normalized in terms of the uniaxial cylinder strength; therefore the formulation can be used for different grades of concrete. The failure criterion cannot be used as plasticity condition, because it describes a boundary for the maximum stresses and does not allow any statements about the plastic deformations.

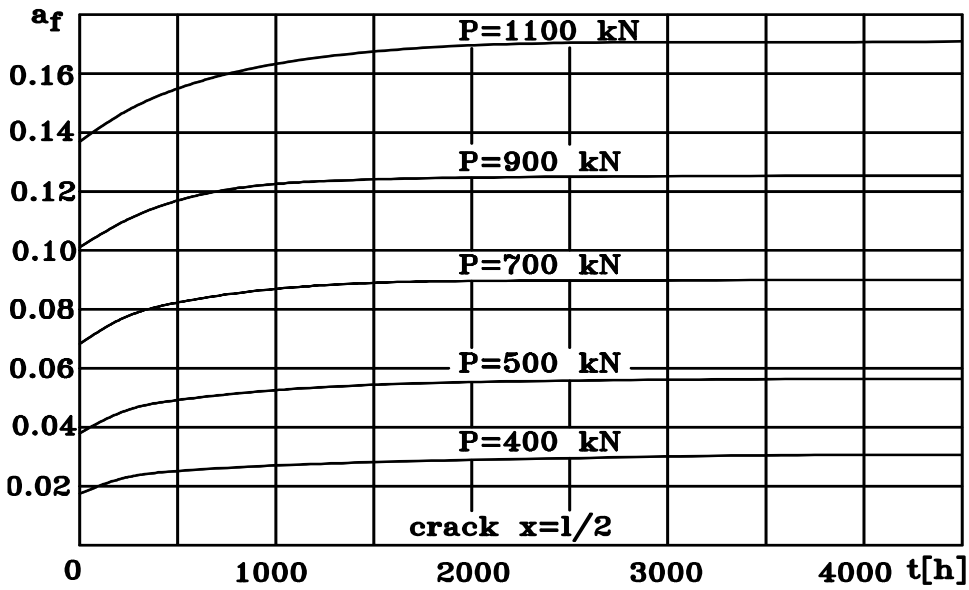

Figure 3. The dependence of the time and loading levels on the crack width a_f for x=l/2

The problem of crack propagation can be solved by evaluating Rice's integral along the contour of the crack top zones, where for different specimens the stress intensity factor could be found (see [13]).

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